SYNTHESIS OF LOSSLESS NETWORKS WITH CONTROLLED SOURCES

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SUMMARY

This thesis reports on a theoretical investigation of lossless networks containing a limited number of controlled sources. The investigation resulted in (a) the derivation of some necessary conditions on the driving-point admittance of a lossless one-port network with one current-controlled voltage source embedded in it and the development of a synthesis procedure for this class of networks when the driving-point admittance is prescribed and satisfies certain sufficient conditions, (b) the development of a synthesis procedure for the lossless two-port network with two current-controlled voltage sources embedded in it when the admittance matrix is prescribed and satisfies certain sufficient conditions. Throughout this thesis it is assumed that the lossless networks without the controlled sources belong to the class of networks that are lumped, linear, finite, passive, bilateral, and time-invariant.

For the lossless one-port network with one current-controlled voltage source embedded in it, an equation for the driving-point admittance is derived in terms of the coefficient of the controlled source and the admittance parameters of a lossless three-port network. From this equation and the necessary and sufficient conditions for lossless networks, some necessary conditions are derived. More specifically, it is shown that for this class of networks the order of any j-axis pole or zero (including a possible pole or zero at infinity) cannot exceed two.

The expression for the driving-point admittance derived from the network and the prescribed driving-point admittance are arranged so that

a set of parameters can be identified for a lossless three-port network in terms of even and odd parts of the numerator and denominator of the prescribed function. Two of the ports are used for the controlled source. The arrangement used for this identification resembles that of the Darlington method for realizing a positive real function as a driving-point admittance. Using the necessary and sufficient conditions for lossless networks, a set of sufficient conditions is stated under which the prescribed function is realizable as a driving-point admittance of a network of this class.

For the lossless N-port network with N current-controlled voltage sources embedded in it, the N x N admittance matrix is expressed in terms of the coefficients of the controlled sources and N x N submatrices of the 3N x 3N admittance matrix of a lossless 3N-port network. As the first step of the synthesis procedure for this class of networks, an equation is given for choosing one of these submatrices when the N x N admittance matrix is prescribed. Equating the prescribed admittance matrix to that derived from the network and making use of the submatrix that has already been chosen give the equation that must be satisfied if the synthesis procedure is to succeed. Separating this equation into even and odd parts allows another submatrix for the lossless 3N-port network to be identified. Also, a product of two submatrices can be identified. With the product of these two submatrices being known, the two submatrices must be found; i.e., the known matrix which has polynomial elements must be factored. Other submatrices of the admittance matrix of the lossless 3N-port network can be chosen freely to facilitate the realization of that network. The connection of the N controlled sources uses 2N of the ports leaving an N-port having the prescribed admittance matrix. The choices of the submatrices discussed above constitute a synthesis procedure if the resulting 3N x 3N admittance matrix is realizable as a lossless 3N-port network. Using the necessary and sufficient conditions for lossless networks, a set of sufficient conditions is stated under which the prescribed N x N admittance matrix is realizable as an N-port network of this class.

The sufficient conditions for the N-port synthesis procedure include the condition that a matrix of polynomials (depending on the prescribed matrix) can be satisfactorily factored. This matrix factorization problem is the major difficulty of the synthesis procedure. A technique exists by which some N x N matrices of polynomials can be factored. This technique makes use of real zeros of the determinant of the given matrix and results in the removal of a matrix whose determinant has real zeros. If the determinant of the given matrix does not have enough real zeros, the method fails.

The special case of N = 2 is considered for the development of a matrix factorization technique using complex zeros. By this technique a matrix whose determinant has a pair of conjugate complex zeros can be removed from the given matrix. No real zeros are required in this procedure. The matrix that is factored out contains elements that are first-degree polynomials or constants while the remaining factor has elements that are one degree lower than those of the given matrix. The sufficient conditions for the removal of a matrix of first degree elements are very lenient. Hence, this technique along with that using real zeros makes it possible to factor most 2×2 matrices.

An extension of the technique mentioned in the above paragraph is considered. It is shown that if a given $N \times N$ matrix of polynomials meets certain conditions, a matrix whose determinant has a pair of conjugate complex zeros can be removed from the given matrix. The matrix that is factored out has first-degree or constant elements in the 2×2 submatrix of the upper left-hand corner and all other elements are constants while the first two columns of the remaining factor have elements that are one degree lower than the elements of the given matrix. By appropriately interchanging columns, this technique can be used to reduce the degree of the elements in columns other than the first and second. Again, the sufficient conditions are very lenient and should allow repeated application in most cases.

As an application of the two-port synthesis procedure, consideration is given to the problem of finding a two-port network to have a prescribed transfer function and a prescribed driving-point admittance when the two-port is terminated in a resistive load. Some sufficient conditions on these two functions are found such that the two-port synthesis procedure presented in this thesis is applicable. The synthesis procedure using two current-controlled voltage sources not only allows power gain but allows the driving-point admittance to be prescribed independently, to some extent, of the transfer admittance.

CHAPTER I

INTRODUCTION

In the past decade, numerous procedures have been developed for the synthesis of active networks to have prescribed admittance matrices. The term "active network" refers to a network that is lumped, linear and finite, but lacks passivity and/or reciprocity. While the case of the lossless network containing positive and negative resistors has been considered, most of the interest in active network theory has been directed toward networks containing positive resistors, positive capacitors and one or more active devices. The active devices usually employed are negative impedance converters, negative impedance inverters, controlled sources, gyrators, or negative resistors. A description of these active devices along with many methods of active network synthesis can be found in a book by Su (1).

Carlin and Youla (2) have investigated the case of the lossless N-port network containing N negative resistors. They have derived the necessary and sufficient conditions for an immittance matrix to be that of a lossless N-port network containing N negative resistors. By combining this result with the necessary and sufficient conditions for the immittance matrix of a lossless N-port network containing N positive resistors they have proved that to any $N \times N$ immittance matrix of real rational functions with arbitrary order and location of zeros and poles, there corresponds an N-port network composed of lossless elements and at most N positive and N negative resistors.

Carlin (3) has derived the necessary and sufficient conditions for realizing a lumped, linear, finite, passive, but nonreciprocal N-port network. He has also derived the necessary and sufficient conditions for the realizability of a lossless, lumped, linear, finite, passive, but nonreciprocal network that possesses an impedance matrix.

Other researchers have used negative resistors along with positive resistors and positive capacitors (this class of networks is referred to as $\pm$ R,C networks). Both Kinariwala (4) and Bello (5) have derived the necessary and sufficient conditions for the realizability of $\pm$ R,C networks.

The work that is most relevant to this investigation is that of Sandberg (6), (7). He has shown that an arbitrary N x N matrix of real rational functions in the complex frequency variable (denoted by s) can be realized as the admittance matrix of a transformerless active RC N-port network containing N real-coefficient controlled sources and cannot, in general, be realized as the admittance matrix of an active RC network containing less than N controlled sources (6). Furthermore, he has shown that an arbitrary symmetric N x N matrix of real rational functions in s can be realized as an unbalanced active RC network requiring no more than N controlled sources (7). The passive RC network required in this realization can always be realized as a (3N + 1)-terminal network of two-terminal impedances with a common reference node and no internal nodes.

This investigation is concerned with the synthesis of lossless networks containing a limited number of controlled sources. Throughout the investigation it is assumed that the lossless networks in which the controlled sources are embedded are lumped, linear, finite, passive, bilateral and time-invariant.

The type of controlled-source that will be considered in this investigation is the current-controlled voltage source. This is a two-port device that is characterized by the following impedance matrix:

$$[z] = \begin{bmatrix} 0 & 0 \\ R & 0 \end{bmatrix}$$

Synthesis procedures are developed for the case of the one-port lossless network containing one current-controlled voltage source and the case of the two-port lossless network containing two current-controlled voltage sources. Included in the two-port synthesis procedure is a new technique for factoring a 2 x 2 matrix having polynomial elements. Examples of the synthesis procedures are presented. Some necessary conditions are derived for the one-port case.

Finally, an application is given and an extension is considered. The two-port synthesis procedure is applied to find a resistively terminated network having a prescribed transfer admittance and a prescribed driving-point admittance. The extension that is considered is that of the synthesis of an N-port lossless network having N current-controlled voltage sources embedded in it.

Some of the terminology and notations that will be used throughout this thesis will be given now.

A rectangular matrix will be denoted by $[A]$ or $[A_{ij}]$ in which A_{ij} denotes the i,j element (the element that appears in the i^{th} row and the j^{th} column) of $[A]$.

A column matrix will be denoted by $A]$ or $A_i]$ in which A_i denotes the i^{th} element of $A]$.

In a partitioned rectangular matrix, the i,j submatrix will be denoted by $[A]_{ij}$.

In a partitioned column matrix, the i^{th} submatrix will be denoted by $A]_i$.

The determinant of a square matrix $[A]$ will be denoted by either det A or $|A|$.

The identity matrix will be denoted by $[U]$.

Admittance parameters of a lossless network will be designated by a bar over the symbol. For example $\bar{y}_{ij}$ is the i,j admittance parameter of a lossless network.

A prescribed function will be designated by a tilde over the symbol. For example, $\tilde{y}$ is a prescribed driving-point admittance.

The absence of a bar or tilde over an admittance symbol will designate an admittance parameter (or driving-point admittance in the one-port case) of the lossless network with controlled sources (or source) embedded in it.

Throughout this thesis, statements regarding poles and zeros include possible poles and zeros at infinity unless the term "finite pole" or "finite zero" is used.

The term "deg p" will be used to mean "the degree of the polynomial p."

The terms "Re w" and "Im w" will be used to denote the real and imaginary parts, respectively, of the complex quantity w.

Additional notation will be introduced and defined as it is needed.

CHAPTER II

THE ONE-PORT NETWORK

In this chapter, the lossless one-port network with one current-controlled voltage source embedded in it will be considered. A network of the type under consideration is shown in Figure 1. First, an equation for the driving-point admittance of the network will be derived. Then, from this equation and the properties of lossless networks, some necessary conditions will be deduced. Finally, a synthesis procedure will be developed, and some sufficient conditions will be given.

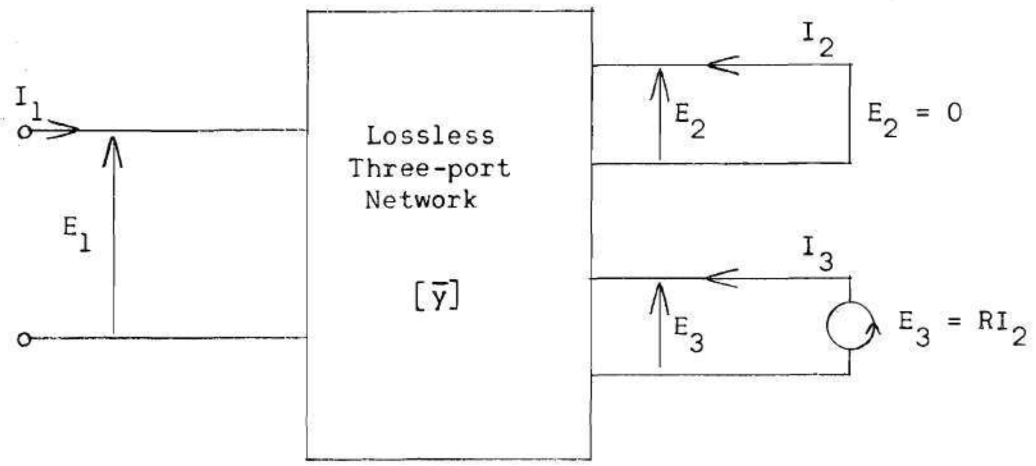

Figure 1. Lossless One-port Network with One Current-controlled Voltage Source Embedded in it.

Derivation of the Driving-point Admittance

A general three-port network whose admittance matrix exists can be described by the following system of equations:

$$\bar{y}_{11}E_1 + \bar{y}_{12}E_2 + \bar{y}_{13}E_3 = I_1$$
$$\bar{y}_{21}E_1 + \bar{y}_{22}E_2 + \bar{y}_{23}E_3 = I_2 \qquad (1)$$
$$\bar{y}_{31}E_1 + \bar{y}_{32}E_2 + \bar{y}_{33}E_3 = I_3$$

The controlled source in the network of Figure 1 imposes the constraints $E_2 = 0$ and $E_3 = RI_2$. Under these conditions, Equations (1) become

$$\bar{y}_{11}E_1 + \bar{y}_{13}RI_2 = I_1$$
$$\bar{y}_{21}E_1 + \bar{y}_{23}RI_2 = I_2 \qquad (2)$$
$$\bar{y}_{31}E_1 + \bar{y}_{33}RI_2 = I_3$$

Let the driving-point admittance of the network of Figure 1 be denoted by y. Solving Equations (2) for I_1/E_1 gives

$$y = \frac{I_1}{E_1} = \bar{y}_{11} + \frac{R\bar{y}_{13}\,\bar{y}_{21}}{1 - R\bar{y}_{23}} \qquad (3)$$

$$= \frac{\bar{y}_{11} - R\bar{y}_{11}\bar{y}_{23} + R\bar{y}_{13}\bar{y}_{21}}{1 - R\bar{y}_{23}} \qquad (4)$$

$$= \bar{y}_{11}\,\frac{\dfrac{R\bar{y}_{13}\bar{y}_{21} - R\bar{y}_{11}\bar{y}_{23}}{\bar{y}_{11}} + 1}{-R\bar{y}_{23} + 1} \qquad (5)$$

Equations (3), (4) and (5) give the driving point admittance of the one-port network of Figure 1 in terms of the parameters of the lossless network and the coefficient of the controlled source. Equation (4) will be used to derive some necessary conditions. The synthesis procedure will be derived by choosing the admittance parameters of the lossless network such that the driving-point admittance given by Equation (5) is the same as the prescribed driving-point admittance.

Some Necessary Conditions

Some necessary conditions on the driving-point admittance will now be derived from Equation (4) and the necessary and sufficient conditions for lossless networks which are given in Appendix I. Since $\bar{y}_{11}$, $\bar{y}_{13}$, $\bar{y}_{21}$ and $\bar{y}_{23}$ are admittance parameters of a lossless network, they are odd rational functions in s.

Let

$$\bar{y}_{11} = \frac{N_{11}}{D} \qquad \bar{y}_{13} = \frac{N_{13}}{D}$$
$$\bar{y}_{21} = \frac{N_{21}}{D} \qquad \bar{y}_{23} = \frac{N_{23}}{D} \tag{6}$$

where N_{11}, N_{13}, N_{21}, N_{23} and D are polynomials in s. If $\bar{y}_{11}$ contains a pole that is not a pole of all of the above parameters, then any parameter not containing that pole will have a common factor in its numerator and denominator. That common factor will not destroy the validity of this discussion and, therefore, there is no loss of generality in assuming that those parameters have a common denominator. Substituting Equations (6) into Equation (4) gives

$$y = \frac{-RN_{11}N_{23} + RN_{13}N_{21} + DN_{11}}{D^2 - RDN_{23}} \tag{7}$$

$$y = \frac{P_1 + Q_1}{P + Q} \tag{8}$$

where $P = D^2$ and $P_1 = -RN_{11}N_{23} + RN_{13}N_{21}$ are even, $Q = -RDN_{23}$ and $Q_1 = DN_{11}$ are odd.

The j-axis poles of y will be considered first. Any finite j-axis pole of y must be a zero of both P and Q. In particular, any such pole must be a zero of $P = D^2$. Since D is a common denominator of the admittance parameters of a lossless network, D has only simple zeros on the j-axis. Therefore, the order of the finite j-axis poles of y cannot exceed two. Equation (3) clearly shows that such a second order pole exists when both $\overline{y}_{13}$ and $\overline{y}_{21}$ contain the same pole and $\overline{y}_{23}$ does not contain that pole. From the necessary and sufficient conditions for the lossless network, it can be seen that neither the degree of P_1 nor that of Q_1 can exceed the degree of P by more than two. Thus, the order of a possible pole of y at infinity cannot exceed two.

Next, the j-axis zeros of y will be considered. Any finite j-axis zero of y must be a zero of both P_1 and Q_1. In particular, any such zero must be a zero of $Q_1 = DN_{11}$. If $\overline{y}_{23}$ has a pole at $s = j\omega_1$ which is not a pole of $\overline{y}_{11}$, then N_{11} and D will contain a common factor $s^2 + \omega_1^2$. Furthermore $\overline{y}_{11}$ may have a simple zero at $s = j\omega_1$. Under these conditions N_{11} would contain the factor $(s^2 + \omega_1^2)^2$ causing $Q_1 = DN_{11}$ to contain the factor $(s^2 + \omega_1^2)^3$. This is the highest order factor that Q_1 could contain (unless it is contained in every term of both numerator and denominator in which case it would have no effect on the number of zeros). Since D would contain the factor $s^2 + \omega_1^2$, $P + Q$ would also contain that factor. So, the maximum degree of a j-axis factor of Q_1 is three and that same j-axis factor occurs in $P + Q$. Therefore, the order of the finite j-axis zeros of y cannot exceed two. Examples can easily be constructed which have a second order zero. One such example is

$$\bar{y}_{11} = \frac{s^2 + 1}{s(s^2 + 2)} = \frac{(s^2 + 1)^2}{s(s^2 + 1)(s^2 + 2)} \qquad \bar{y}_{13} = \frac{1}{s} = \frac{(s^2+1)(s^2+2)}{s(s^2+1)(s^2+2)}$$

$$\bar{y}_{21} = \frac{1}{s} = \frac{(s^2 + 1)(s^2 + 2)}{s(s^2 + 1)(s^2 + 2)} \qquad \bar{y}_{23} = \frac{1}{s(s^2 + 1)(s^2 + 2)}$$

With appropriately chosen functions for $\bar{y}_{22}$ and $\bar{y}_{33}$, the above functions are realizable as admittance parameters of a lossless network. Using the above parameters and choosing $R = 1$, Equation (7) becomes

$$y = \frac{-(s^2 + 1)^2 + (s^2 + 1)^2(s^2 + 2)^2 + s(s^2 + 1)^3(s^2 + 2)}{s^2(s^2 + 1)^2(s^2 + 2)^2 - s(s^2 + 1)(s^2 + 2)}$$

$$= \frac{(s^2 + 1)^2(s^3 + s^2 + 2s + 3)}{s(s^2 + 2)(s^5 + 3s^3 + 2s - 1)}$$

In the above example, y has a second order zero at $s = \pm j1$. To see that the order of a possible zero at infinity cannot exceed two, it is only necessary to notice that the degree of P can exceed that of Q_1 by no more than one and the degree of Q can exceed that of Q_1 by no more than two.

The above discussion on the order of the j-axis poles and zeros of y can be summarized by the following theorem:

Theorem 1

If a three-port lossless network whose admittance matrix exists has a current-controlled voltage source connected between ports 2 and 3, then the driving-point admittance y at port 1 is a real rational function in s that satisfies the two conditions:

1. The order of any j-axis pole of y does not exceed two.
2. The order of any j-axis zero of y does not exceed two.

The Synthesis Procedure

Let the prescribed driving-point admittance be

$$\tilde{y} = \frac{m_1 + n_1}{m + n} \tag{9}$$

where m and m_1 are even polynomials and n and n_1 are odd polynomials in s. Case A of the synthesis procedure will be derived in detail and only the final equations given for Case B.

Case A

First, Equation (9) should be arranged in the form

$$\tilde{y} = \frac{m_1}{n} \; \frac{\frac{n_1}{m_1} + 1}{\frac{m}{n} + 1} \tag{10}$$

Now, equating the corresponding parts of the right-hand members of Equations (5) and (10) gives

$$\bar{y}_{11} = \frac{m_1}{n} \tag{11}$$

$$-R\bar{y}_{23} = \frac{m}{n} \tag{12}$$

$$\frac{R\bar{y}_{13}\bar{y}_{21} - R\bar{y}_{11}\bar{y}_{23}}{\bar{y}_{11}} = \frac{n_1}{m_1} \tag{13}$$

Substituting Equations (11) and (12) into Equation (13) gives

$$\frac{R\bar{y}_{13}\bar{y}_{21} + \frac{m_1}{n}\frac{m}{n}}{\frac{m_1}{n}} = \frac{n_1}{m_1}$$

from which

$$R\bar{y}_{13}\bar{y}_{21} = \frac{nn_1 - mm_1}{n^2} \tag{14}$$

If admittance parameters satisfying Equations (11), (12) and (14) can be realized by a lossless three-port network, the synthesis procedure will be successful. Equations (11) and (12) specify $\bar{y}_{11}$ and $\bar{y}_{23}$ precisely while Equation (14) specifies the product $\bar{y}_{13}\bar{y}_{21}$. To find $\bar{y}_{13}$ and $\bar{y}_{21}$, $nn_1 - mm_1$ is factored and some of the factors are alloted to $\bar{y}_{13}$ and the balance to $\bar{y}_{21}$. It is unimportant which factors are allotted to $\bar{y}_{13}$ and which to $\bar{y}_{21}$ so long as sufficient conditions for the realization of the three-port lossless network can be maintained. The parameters $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen freely to facilitate the realization. Finally, the current-controlled voltage source is connected between ports 2 and 3 of the lossless network as indicated in Figure 1. With the above choice of admittance parameters and with the controlled source connected, the driving-point admittance which is given by Equation (5) for the network is equal to that prescribed by Equation (9).

Case B

For this case similar identifications can be made between Equation (5) and the following arrangement of Equation (9):

$$\tilde{y} = \frac{n_1}{m} \frac{\frac{m_1}{n_1} + 1}{\frac{n}{m} + 1} \tag{15}$$

The equations specifying the admittance parameters for the lossless network are

$$\bar{y}_{11} = \frac{n_1}{m} \tag{16}$$

$$-R\bar{y}_{23} = \frac{n}{m} \tag{17}$$

$$R\bar{y}_{13}\bar{y}_{21} = \frac{mm_1 - nn_1}{m^2} \tag{18}$$

Again, $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen freely to facilitate the realization.

For both Case A and Case B, the parameters $\bar{y}_{22}$ and $\bar{y}_{33}$ are chosen such that any pole of $\bar{y}_{13}$, $\bar{y}_{21}$ or $\bar{y}_{23}$ is also a pole of $\bar{y}_{22}$ and $\bar{y}_{33}$. Furthermore, the residues in the poles of $\bar{y}_{22}$ and $\bar{y}_{33}$ are chosen large enough such that the residue condition is satisfied in every pole. The conditions under which the synthesis procedures for Case A and Case B are successful are given in the following section.

Some Sufficient Conditions

The problem now is to determine some conditions on the prescribed driving-point admittance $\tilde{y}$ that are sufficient for the synthesis procedure of the previous section to be successful. This can be done by considering Equations (11), (12) and (14) for Case A and Equations (16), (17) and (18) for Case B along with the necessary and sufficient conditions for lossless networks which are given in Appendix I. The conditions will be stated as Theorem 2 with the proof of sufficiency following the statement of the theorem.

Theorem 2

Let a prescribed real ration function be $\tilde{y} = \frac{m_1 + n_1}{m + n}$ where m and m_1 are even polynomials and n and n_1 are odd polynomials in s. The function $\tilde{y}$ can be realized as a driving-point admittance of a lossless network with one current-controlled voltage source embedded in it if the conditions given under Case A or Case B below are satisfied.

Case A

1. m_1/n is a reactance function.

2. m/n has only j-axis poles and those are simple.

3. $(nn_1 - mm_1)/n^2$ has only j-axis poles and the order of those poles does not exceed two.

4. Any simple or second order pole of $(nn_1 - mm_1)/n^2$ is a simple pole of m_1/n.

5. If m_1/n has a zero at infinity, then $(nn_1 - mm_1)/n^2$ has a zero there (second order or higher).

6. If m_1/n has a zero at the origin, then $(nn_1 - mm_1)/n^2$ has a zero there (second order or higher).

7. The number of quadrantal zeros of $(nn_1 - mm_1)/n^2$ does not

exceed $8[\frac{\delta + \varepsilon}{4}]$* where δ is the number of finite poles of m_1/n and

$\varepsilon = 1$ if δ is odd and m_1/n has a pole at infinity.

$= 0$ if δ is odd and m_1/n has a zero at infinity.

$= 0$ if δ is even and m_1/n has a pole at infinity.

$= -1$ if δ is even and m_1/n has a zero at infinity.

Case B

1. n_1/m is a reactance function.

2. n/m has only j-axis poles and those are simple.

3. $(mm_1 - nn_1)/m^2$ has only j-axis poles and the order of those poles does not exceed two.

4. Any simple or second order pole of $(mm_1 - nn_1)/m^2$ is a simple pole of n_1/m.

5. If n_1/m has a zero at infinity, then $(mm_1 - nn_1)/m^2$ has a zero there (second order or higher).

6. If n_1/m has a zero at the origin, then $(mm_1 - nn_1)/m^2$ has a zero there (second order or higher).

7. The number of quadrantal zeros of $(mm_1 - nn_1)/m^2$ does not exceed $8[\frac{\delta + \varepsilon}{4}]$ where δ is the number of finite poles in n_1/m and

$\varepsilon = 1$ if δ is odd and n_1/m has a pole at infinity.

$= 0$ if δ is odd and n_1/m has a zero at infinity.

$= 0$ if δ is even and n_1/m has a pole at infinity.

$= -1$ if δ is even and n_1/m has a zero at infinity.

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*Brackets [ ] used with the quantity $\frac{\delta + \varepsilon}{4}$ indicate the largest integer that is equal to or less than $\frac{\delta + \varepsilon}{4}$.

Proof

The proof will be given for Case A. That for Case B is identical except that m and m_1 are interchanged and n and n_1 are interchanged.

Condition 1 ensures that $\bar{y}_{11}$ is a reactance function. This is the only requirement on $\bar{y}_{11}$ in order for the 3 x 3 admittance matrix to be realizable by a three-port lossless network.

Condition 2 ensures that $\bar{y}_{23}$ has only j-axis poles and that those are simple. Since $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen freely, they can be chosen such that any pole of $\bar{y}_{23}$ is a pole of $\bar{y}_{22}$ and $\bar{y}_{33}$. Furthermore, the residues in the poles of $\bar{y}_{22}$ and $\bar{y}_{33}$ and the coefficient R can be chosen sufficiently large such that the residue condition is satisfied for all poles. The parameters $\bar{y}_{13}$ and $\bar{y}_{21}$ may require that $\bar{y}_{22}$ and $\bar{y}_{33}$ have additional poles.

Conditions 3 and 4 ensure that $(nn_1 - mm_1)/n^2$ can be written as

$$\frac{nn_1-mm_1}{n^2} = \frac{(s^2+\omega_1^2)(s^2+\omega_2^2)\ldots(s^2+\omega_p^2)(s^4+b_1s^2+c_1)(s^4+b_2s^2+c_2)\ldots(s^4+b_qs^2+c_q)}{D^2} \tag{19}$$

where D is a polynomial whose zeros are the finite poles of $\bar{y}_{11}$. The degree of D is δ. There are no restrictions on $\omega_1, \omega_2, \ldots, \omega_p$. Also some of the numerator factors $s^2 + \omega_i^2$ may be factors of D. A factor of the form $s^4 + b_js^2 + c_j$ with $b_j^2 < 4c_j$ represents a set of quadrantal zeros of $(nn_1 - mm_1)/n^2$. Condition 3 further ensures that the degree of the numerator of the right-hand side of Equation (19) does not exceed $2\delta + 2$.

Assume now that Equation (19) has no quadrantal zeros. The case of quadrantal zeros will be considered later. Under the assumption of no quadrantal zeros, the four possible situations that can occur will be considered:

1. If δ is odd and m_1/n has a pole at infinity, the numerator of the right-hand member of Equation (19) has at most $\delta + 1$ factors, all being of the form $s^2 + \omega_i^2$. Allotting $(\delta + 1)/2$ of these factors to N_{13} leaves at most $(\delta + 1)/2$ factors for N_{21}. Thus, the degree of neither N_{13} nor N_{21} exceeds $\delta + 1$. The definitions $\bar{y}_{13} = N_{13}/D$ and $\bar{y}_{21} = N_{21}/D$ give satisfactory parameters since $\bar{y}_{11}$ has a pole at infinity. In this discussion, a satisfactory or acceptable function for $\bar{y}_{13}$ or $\bar{y}_{21}$ is a function that has only simple j-axis poles and, furthermore, any pole of $\bar{y}_{13}$ or $\bar{y}_{21}$ is a pole of $\bar{y}_{11}$.

2. If δ is odd and m_1/n has a zero at infinity, Condition 5 allows the numerator of the right-hand side of Equation (19) to have no more than $\delta - 1$ factors. Allotting $(\delta - 1)/2$ of these factors to N_{13} leaves at most $(\delta - 1)/2$ factors for N_{21}. Thus, the degree of neither N_{13} nor N_{21} exceeds $\delta - 1$. This means that the functions $\bar{y}_{13}$ and $\bar{y}_{21}$ have zeros at infinity, which is a satisfactory condition.

3. If δ is even and m_1/n has a pole at infinity, Condition 6 requires that at least one of the ω_i of Equation (19), say ω_1, be zero. Since Condition 3 limits the total number of numerator factors to $\delta + 1$ there are no more than δ factors in the product $(s^2 + \omega_2^2)(s^2 + \omega_3^2)\ldots(s^2 + \omega_p^2)$. Allotting $\delta/2$ of these factors along with the factor s to N_{13} leaves no more than $\delta/2$ factors of the form $s^2 + \omega_i^2$ along with the factor s for N_{21}. Thus, the degree of neither N_{13} nor N_{21} exceed $\delta + 1$. This gives satisfactory

functions for $\bar{y}_{13}$ and $\bar{y}_{21}$ since $\bar{y}_{11}$ has a pole at infinity.

4. If δ is even and m_1/n has a zero at infinity, Condition 5 limits the total number of factors in the numerator of Equation (19) to $\delta - 1$. As in situation 3, the first of these is s^2, so there are no more than $\delta - 2$ factors in the product $(s^2 + \omega_2^2)(s^2 + \omega_3^2)\ldots(s^2 + \omega_p^2)$. Allotting $(\delta - 2)/2$ of these factors along with s to N_{13} leaves no more than $(\delta - 2)/2$ second-degree factors along with s for N_{21}. Thus, the degree of neither N_{13} nor N_{21} exceeds $\delta - 1$. This means that the functions $\bar{y}_{13}$ and $\bar{y}_{21}$ have zeros at infinity, which is a satisfactory condition.

In situations 1 and 2 above, if the total number of factors in the numerator of the right-hand side of Equation (19) is less than $(\delta + 1)/2$ or $(\delta - 1)/2$ respectively, then obviously all of the factors could be allotted to N_{13} and N_{21} set equal to a constant. Similarly, in situations 3 and 4, if the number of factors in the product $(s^2 + \omega_2^2)(s^2 + \omega_3^2)\ldots(s^2 + \omega_p^2)$ is less than $\delta/2$ or $(\delta - 2)/2$ respectively, then all of these factors along with s could be allotted to N_{13} and s alone allotted to N_{21}.

The above reasoning shows that for any situation in which the function $(nn_1 - mm_1)/n^2$ has no quadrantal zeros, that function can be factored into two functions $\bar{y}_{13}$ and $\bar{y}_{21}$ such that their poles are simple and on the j-axis and such that any pole of $\bar{y}_{13}$ or $\bar{y}_{21}$ is a pole of $\bar{y}_{11}$. Since $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen freely, they can easily be chosen such that any pole of $\bar{y}_{13}$, $\bar{y}_{21}$ or $\bar{y}_{23}$ is a simple pole of $\bar{y}_{22}$ and $\bar{y}_{33}$. Furthermore, the residues in the poles of $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen sufficiently large such that the residue condition (Condition

3 in Appendix I) is satisfied for all poles of the elements of the 3 x 3 admittance matrix. Therefore, Conditions 1 through 6 ensure the realizability of the admittance parameters specified by the synthesis procedure when $(nn_1 - mm_1)/n^2$ has no quadrantal zeros.

If $(nn_1 - mm_1)/n^2$ has quadrantal zeros, each set of quadrantal zeros replaces a pair of the second degree factors of N_{13} or a pair of those factors of N_{21}. A check of the four possible situations listed in Condition 7 of the theorem shows that $\bar{y}_{13}$ and $\bar{y}_{21}$ are acceptable functions if the number of quadrantal zeros in either does not exceed $4[\frac{\delta + \varepsilon}{4}]$. Thus, Condition 7 limits the number of quadrantal zeros in $(nn_1 - mm_1)/n^2$ to a sufficiently small number so that it will never be necessary to allot part of a quadrantal factor to N_{13} and part to N_{21}. Now, Conditions 3, 4, 5 and 6 ensure that any additional factors (these will be second degree) can be allotted to N_{13} and N_{21} to give acceptable functions for $\bar{y}_{13}$ and $\bar{y}_{21}$. The parameters $\bar{y}_{22}$ and $\bar{y}_{33}$ are chosen as before. Therefore, if the prescribed function $\tilde{y}$ satisfies Conditions 1 through 7, the synthesis procedure for Case A or Case B will be successful. This completes the proof of Theorem 2.

An Example

As an example of the synthesis procedure, a lossless network containing one current-controlled voltage source will be found having the prescribed driving-point admittance

$$\tilde{y} = \frac{m_1 + n_1}{m + n} = \frac{s^2 - 2s + 1}{s^3 + s^2 + 2s + 10} \tag{20}$$

Equations (11), (12) and (14) give

$$\bar{y}_{11} = \frac{m_1}{n} = \frac{s^2 + 1}{s(s^2 + 2)}$$

$$\bar{y}_{23} = -\frac{1}{R}\frac{m}{n} = -\frac{1}{R}\frac{s^2 + 10}{s(s^2 + 2)}$$

$$\bar{y}_{13}\bar{y}_{21} = \frac{1}{R}\frac{nn_1 - mm_1}{n^2} = \frac{1}{R}\frac{(s^2 + 2s)(-2s) - (s^2 + 10)(s^2 + 1)}{s^2(s^2 + 2)^2}$$

$$= -\frac{3}{R}\frac{(s^2 + 4.21)(s^2 + 0.79)}{s^2(s^2 + 2)^2}$$

Let

$$\bar{y}_{13} = -\sqrt{\frac{3}{R}}\frac{s^2 + 0.79}{s(s^2 + 2)} \qquad \bar{y}_{21} = \sqrt{\frac{3}{R}}\frac{s^2 + 4.21}{s(s^2 + 2)}$$

Since the residues for the functions $\bar{y}_{22}$ and $\bar{y}_{33}$ can be chosen arbitrarily large, any nonzero value of R is satisfactory. The choice of a very small value for R will give excessively large residue values for $\bar{y}_{22}$ and $\bar{y}_{33}$. In some cases it may be desirable to avoid the use of transformers in the realization. In this case R should be chosen sufficiently large such that the residue matrix in each pole is dominant (8). A symmetric matrix of real constants is said to be a dominant matrix if each of its main-diagonal elements is not less than the sum of the absolute values of all the other elements in the same row. For this example, R will be chosen to be 100. The admittance matrix for the lossless three-port network is

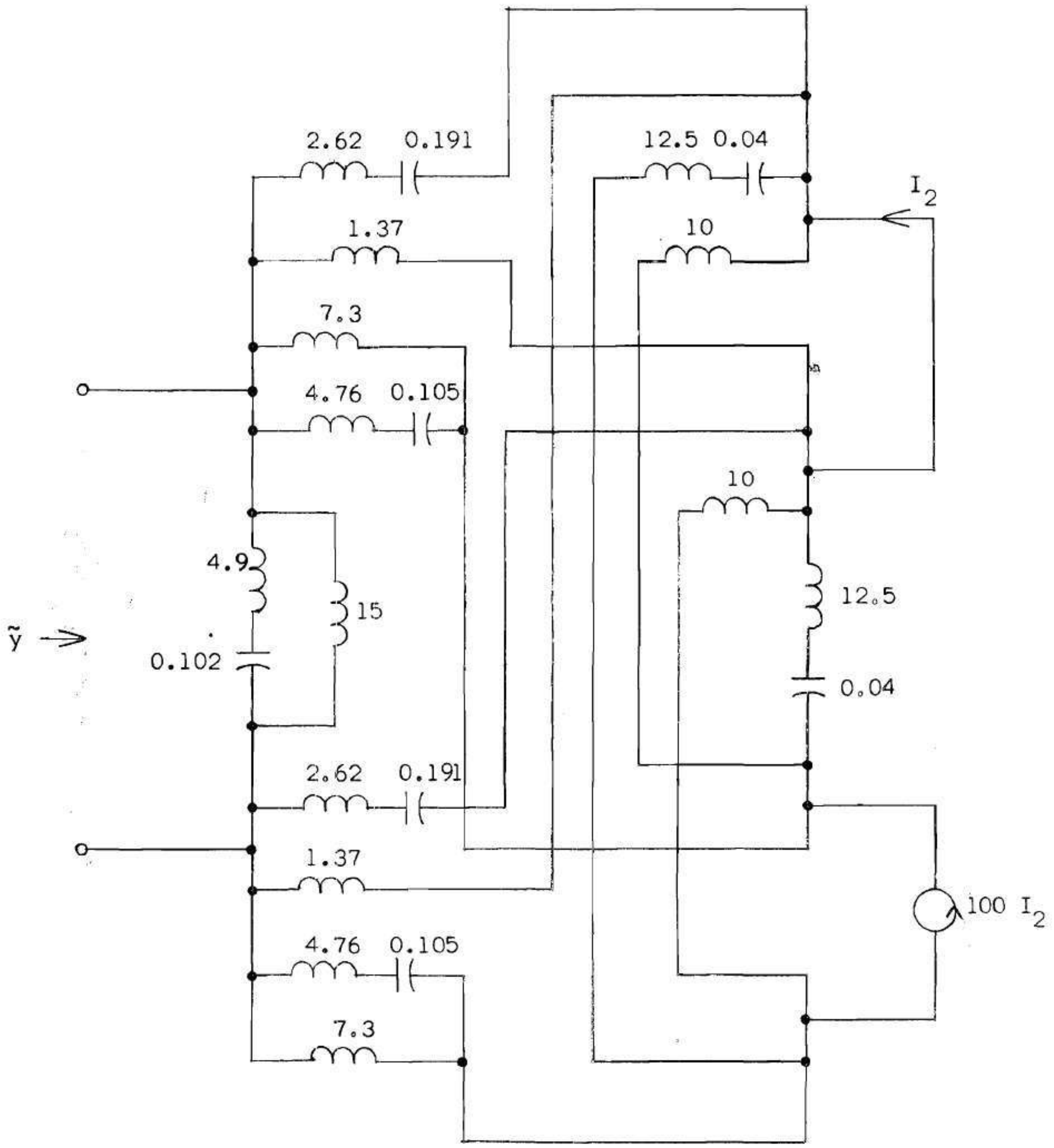
$$[y] = \begin{bmatrix} \frac{s^2 + 1}{s(s^2 + 2)} & 0.1732\,\frac{s^2 + 4.21}{s(s^2 + 2)} & -0.1732\,\frac{s^2 + 0.79}{s(s^2 + 2)} \\ 0.1732\,\frac{s^2 + 4.21}{s(s^2 + 2)} & y_{22} & -0.01\,\frac{s^2 + 10}{s(s^2 + 2)} \\ -0.1732\,\frac{s^2 + 0.79}{s(s^2 + 2)} & -0.01\,\frac{s^2 + 10}{s(s^2 + 2)} & y_{33} \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0.365 & -0.0684 \\ 0.365 & k_{22}^{(o)} & -0.05 \\ -0.0684 & -0.05 & k_{33}^{(o)} \end{bmatrix} + \frac{s}{s^2 + 2}\begin{bmatrix} 0.5 & -0.191 & -0.105 \\ -0.191 & 2k_{22}^{(1)} & 0.04 \\ -0.105 & 0.04 & 2k_{33}^{(1)} \end{bmatrix} \quad (21)$$

No values need be chosen for the residues of the functions $\bar{y}_{22}$ and $\bar{y}_{33}$. They will be allowed to have values such that for rows 2 and 3 the residue matrices of Equation (21) satisfy the dominant condition with the equal sign. The complete network for this realization is shown in Figure 2.

Summary

In this chapter some necessary conditions have been derived for the lossless one-port network having one current-controlled voltage source embedded in it. A synthesis procedure for this class of networks and some sufficient conditions have been derived. An example has been worked illustrating the synthesis procedure.



Henrys, Farads

Figure 2. Network Realizing the Driving-point Admittance of Equation (20).

CHAPTER III

THE TWO-PORT NETWORK

In this chapter, the class of networks of primary interest is that of the lossless two-port network with two current-controlled voltage sources embedded in it. However, since the derivation of the admittance matrix for the case of an N-port network with N controlled sources embedded in it is identical to that for the case of a two-port network with two controlled sources embedded in it, the derivation will be carried out for the N-port case. After an equation for the admittance matrix of the network of Figure 3 has been derived, a synthesis procedure will be developed and some sufficient conditions will be stated. The synthesis procedure requires that a matrix of polynomial elements be factored. At that point, because of the difficulty of the matrix factorization step, the procedure will be limited to the special case of $N = 2$. A new technique will be developed for factoring a certain class of matrices. This new technique, when used along with an existing technique, will make possible the factorization of most 2×2 matrices having polynomial elements.

Derivation of the Admittance Matrix

A general 3N-port network whose admittance matrix exists can be described by the following matrix equation:

$$[\,\bar{y}\,]\, E] = I] \tag{22}$$

where $[\bar{y}]$ is the $3N \times 3N$ admittance matrix of the 3N-port network,

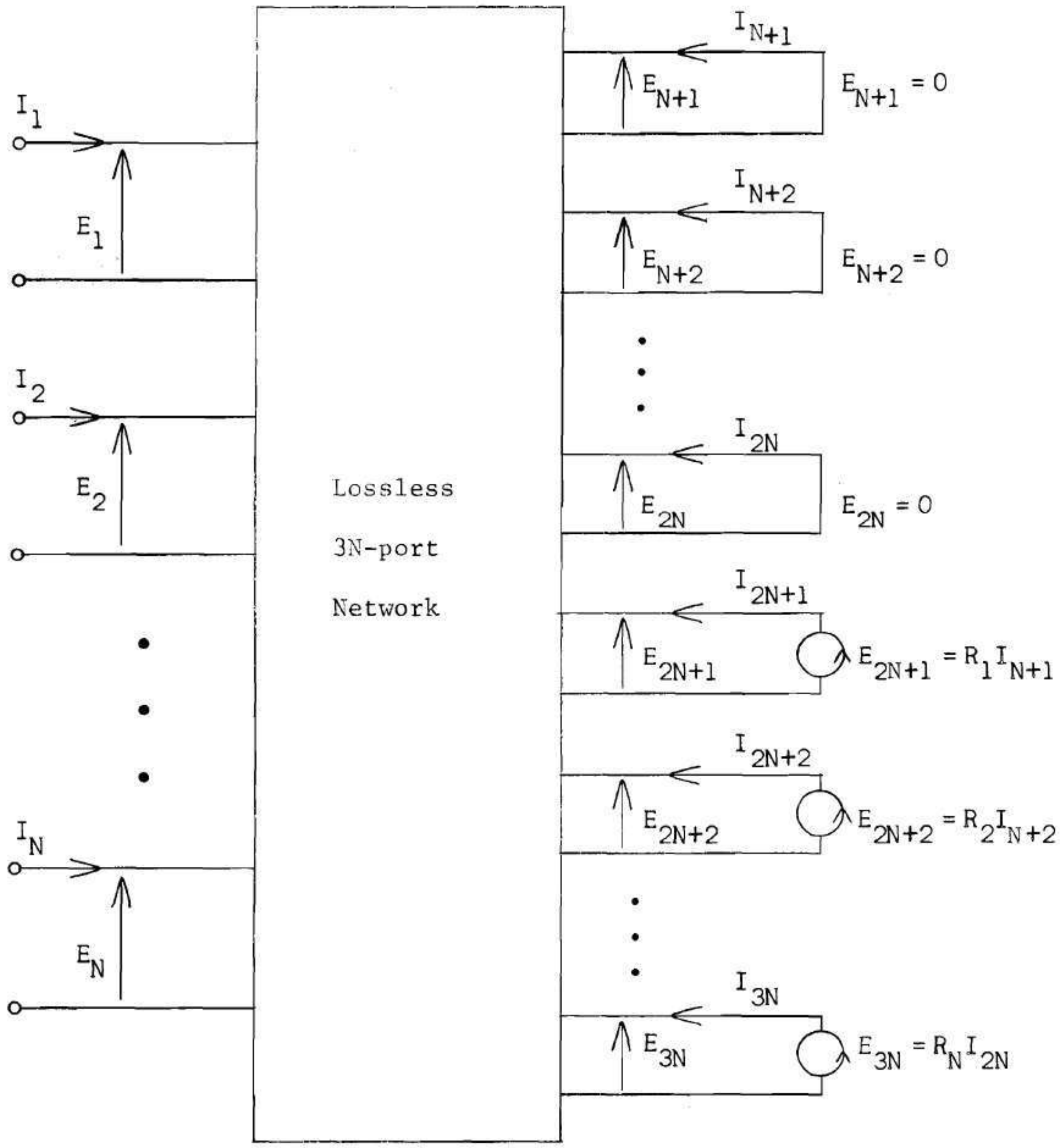


Figure 3. Lossless N-port Network Containing N Current-controlled Voltage Sources.

$E]$ and $I]$ are $3N \times 1$ matrices with E_j and I_i defined at the ports of the $3N$-port network as indicated in Figure 3 for $i, j = 1, 2, \ldots, 3N$. Let the admittance matrix $[\bar{y}]$ be partitioned into nine $N \times N$ submatrices as indicated by Equation (23) and let each of the column matrices $E]$ and $I]$ be partitioned into three $N \times 1$ submatrices as indicated by Equations (24)

$$[\bar{y}] = \begin{bmatrix} [\bar{y}]_{11} & [\bar{y}]_{12} & [\bar{y}]_{13} \\ [\bar{y}]_{21} & [\bar{y}]_{22} & [\bar{y}]_{23} \\ [\bar{y}]_{31} & [\bar{y}]_{32} & [\bar{y}]_{33} \end{bmatrix} \tag{23}$$

$$E] = \left.\begin{matrix} E]_1 \\ E]_2 \\ E]_3 \end{matrix}\right] \qquad I] = \left.\begin{matrix} I]_1 \\ I]_2 \\ I]_3 \end{matrix}\right] \tag{24}$$

Using Equations (23) and (24), Equation (22) becomes

$$\begin{aligned} [y]_{11}\, E]_1 + [\bar{y}]_{12}\, E]_2 + [\bar{y}]_{13}\, E]_3 &= I]_1 \\ [\bar{y}]_{21}\, E]_1 + [\bar{y}]_{22}\, E]_2 + [\bar{y}]_{23}\, E]_3 &= I]_2 \\ [\bar{y}]_{31}\, E]_1 + [\bar{y}]_{32}\, E]_2 + [\bar{y}]_{33}\, E]_3 &= I]_3 \end{aligned} \tag{25}$$

The controlled sources in the network of Figure 3 impose the constraints

$$E]_2 = 0] \qquad E]_3 = [R]\, I]_2 \tag{26}$$

where $0]$ is the $N \times 1$ zero matrix and $[R]$ is the diagonal matrix having elements $R_1, R_2, \ldots, R_N$. Under the conditions imposed by the controlled sources, Equations (25) become

$$\begin{aligned}
[\bar{y}]_{11}\, E]_1 + [\bar{y}]_{13}\, [R]\, I]_2 &= I]_1 \\
[\bar{y}]_{21}\, E]_1 + [\bar{y}]_{23}\, [R]\, I]_2 &= I]_2 \\
[\bar{y}]_{31}\, E]_1 + [\bar{y}]_{33}\, [R]\, I]_2 &= I]_3
\end{aligned} \tag{27}$$

Solving the second of Equations (27) for $I]_2$ gives

$$I]_2 = -\left[[\bar{y}]_{23}\, [R] - [U]\right]^{-1} [\bar{y}]_{21}\, E]_1 \tag{28}$$

Substituting Equation (28) into the first of Equations (27) gives

$$[\bar{y}]_{11}\, E]_1 - [\bar{y}]_{13}\, [R]\left[[\bar{y}]_{23}\, [R] - [U]\right]^{-1} [\bar{y}]_{21}\Big] E]_1 = I]_1$$

On rearranging, this equation becomes

$$\left[[\bar{y}]_{11} - [\bar{y}]_{13}\, [R]\left[[\bar{y}]_{23}\, [R] - [U]\right]^{-1} [\bar{y}]_{21}\right] E]_1 = I]_1 \tag{29}$$

From Equation (29) the admittance matrix of the N-port network with N current-controlled voltage sources embedded in it can be identified as

$$[y] = [\bar{y}]_{11} - [\bar{y}]_{13}[R]\left[[\bar{y}]_{23}[R] - [U]\right]^{-1} [\bar{y}]_{21} \tag{30}$$

Equation (30) forms the basis for the synthesis procedure that follows.

The Synthesis Procedure

Let the prescribed admittance matrix be

$$[\tilde{y}] = \frac{[m_{ij} + n_{ij}]}{m + n} \tag{31}$$

where $m + n$ is a common denominator for the elements of the prescribed admittance matrix $[\tilde{y}]$, m and m_{ij} are even polynomials and n and n_{ij} are odd polynomials in s for $i,j = 1,2,\ldots,N$.

The synthesis procedure will be derived by forcing the admittance matrix derived from the network to be equal to the prescribed admittance matrix and then choosing appropriate functions for the admittance parameters of the lossless 3N-port network. To this end, the right-hand member of Equation (30) is equated to that of Equation (31). This gives

$$[y]_{11} - [\bar{y}]_{13}[R]\left[[\bar{y}]_{23}\,[R] - [U]\right]^{-1}[\bar{y}]_{21} = \frac{[m_{ij} + n_{ij}]}{m + n} \tag{32}$$

As in the one-port case, the synthesis procedure for the N-port case can be derived for either Case A or Case B. The derivation will be carried out in detail for Case A and only the final equations given for Case B.

<u>Case A</u>

Let

$$[y]_{11} = \frac{[m_{ij}]}{n} \tag{33}$$

Substituting Equation (33) into Equation (32) gives

$$-[\bar{y}]_{13}[R]\left[[\bar{y}]_{23}\,[R] - [U]\right]^{-1}[\bar{y}]_{21} = \frac{[m_{ij} + n_{ij}]}{m + n} - \frac{[m_{ij}]}{n}$$

$$= \frac{[nn_{ij} - mm_{ij}]}{n(m + n)}$$

On taking the inverse of both sides, this equation becomes

$$-[\bar{y}]_{21}^{-1}\Big[[\bar{y}]_{23}[R] - [U]\Big][R]^{-1}[\bar{y}]_{13}^{-1} = n(m+n)[nn_{ij} - mm_{ij}]^{-1} \tag{34}$$

Equating the even part of the left-hand member to that of the right-hand member in Equation (34) gives

$$[\bar{y}]_{21}^{-1}\ [R]^{-1}[\bar{y}]_{13}^{-1} = n^2[nn_{ij} - mm_{ij}]^{-1} \tag{35}$$

On taking the inverse of both sides, Equation (35) becomes

$$[\bar{y}]_{13}\ [R]\ [\bar{y}]_{21} = \frac{[nn_{ij} - mm_{ij}]}{n^2} \tag{36}$$

Let $[R]_1$ and $[R]_2$ be two N x N diagonal matrices such that

$$[R] = [R]_1\ [R]_2 \tag{37}$$

Substituting Equation (37) into Equation (36) gives

$$[\bar{y}]_{13}\ [R]_1[R]_2[\bar{y}]_{21} = \frac{[nn_{ij} - mm_{ij}]}{n^2} \tag{38}$$

Now, returning to Equation (34) and equating the odd part of its left-hand member to that of its right-hand member give

$$\begin{aligned} -[\bar{y}]_{21}^{-1}\ [\bar{y}]_{23}[\bar{y}]_{13}^{-1} &= mn[nn_{ij} - mm_{ij}]^{-1} \\ &= \frac{m}{n}\ n^2\ [nn_{ij} - mm_{ij}]^{-1} \end{aligned} \tag{39}$$

Substituting Equation (35) into Equation (39) gives

$$-[\bar{y}]_{21}^{-1}\ [\bar{y}]_{23}[\bar{y}]_{13}^{-1} = \frac{m}{n}\ [\bar{y}]_{21}^{-1}\ [R]^{-1}\ [\bar{y}]_{13}^{-1}$$

Premultiplying by $-[\bar{y}]_{21}$ and postmultiplying by $[\bar{y}]_{13}$ give

$$[\bar{y}]_{23} = -\frac{m}{n}[R]^{-1} \tag{40}$$

If submatrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ can be found such that Equation (38) is satisfied and if these submatrices along with $[\bar{y}]_{11}$ and $[\bar{y}]_{23}$ (as specified by Equations (33) and (40) respectively) are realizable by a lossless 3N-port network, then the N-port synthesis procedure will be successful. A key step in the N-port synthesis is the matrix factorization; i.e., finding $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ such that Equation (38) is satisfied. This step is considered later in this chapter for 2 x 2 matrices and in Chapter IV and Appendix II for N x N matrices. The submatrices $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ can be chosen freely to facilitate the realization.

The N-port synthesis procedure for Case A can be summarized as follows:

1. Choose $[\bar{y}]_{11}$ in accordance with Equation (33).

2. Find matrices $[\bar{y}]_{13}[R]_1$ and $[R]_2[\bar{y}]_{21}$ such that Equation (32) is satisfied.

3. To finish specifying $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$, pick $[R]_1$ and $[R]_2$. This also specifies $[R]$ through Equation (37).

4. The submatrix $[\bar{y}]_{23}$ is now given by Equation (40).

5. Choose $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$.

6. Realize the 3N-port network and connect the current-controlled voltage sources as indicated in Figure 3. The coefficients of these sources are the elements of $[R]$.

When the above steps are followed, Equation (32) is satisfied and the synthesis procedure is completed for Case A.

Case B

The synthesis procedure for Case B is the same as that for Case A except that m and n are interchanged and m_{ij} and n_{ij} are interchanged. The equations for Case B, which correspond to Equations (33), (38) and (40) for Case A, are:

$$[\bar{y}]_{11} = \frac{[n_{ij}]}{m} \tag{41}$$

$$[\bar{y}]_{13}[R]_1[R]_2[\bar{y}]_{21} = \frac{[mm_{ij} - nn_{ij}]}{m^2} \tag{42}$$

$$[\bar{y}]_{23} = -\frac{n}{m}[R]^{-1} \tag{43}$$

In this section, procedures have been presented for specifying a 3N x 3N admittance matrix $[\bar{y}]$ for the lossless network of Figure 3 when the N x N admittance matrix $[\tilde{y}]$ for the overall network is prescribed. Of course, the procedure for choosing $[\bar{y}]$ was devised such that $[\bar{y}]$ would have the salient features of a lossless network. More specifically, for Case A or Case B, the elements of $[\bar{y}]_{11}$, $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ are odd rational functions; the elements of the product $[\bar{y}]_{13}[R]_1$ $[R]_2[y]_{21}$ are even function, allowing the possibility of finding $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ with odd elements. Conditions that ensure the realizability of a lossless network corresponding to $[\bar{y}]$, and thus assure the success of the synthesis procedure, are given in the following section.

Some Sufficient Conditions

The purpose of this section is to present some conditions on the prescribed admittance matrix $[\tilde{y}]$ that are sufficient for the synthesis procedure of the previous section to be successful. Some sufficient conditions are suggested by consideration of Equations (33), (38) and (40) for Case A and Equations (41), (42) and (43) for Case B along with the necessary and sufficient conditions for lossless networks (given in Appendix I). These conditions are stated as Theorem 3 with the proof of sufficiency following the statement of the theorem.

Theorem 3

Let a prescribed matrix of real rational functions be $[\tilde{y}] = [m_{ij} + n_{ij}]/(m + n)$ where $m + n$ is a common denominator for the elements of $[\tilde{y}]$, m and m_{ij} are even polynomials, n and n_{ij} are odd polynomials in s and $i,j = 1,2,\ldots,N$. The matrix $[\tilde{y}]$ can be realized as the admittance matrix of a lossless N-port network with N current-controlled voltage sources embedded in it if the conditions given under Case A or Case B below are satisfied.

Case A

1. $[m_{ij}]/n$ is realizable as a lossless N-port network and, further, the residue matrix is positive definite at each pole.

2. m/n has only j-axis poles and those are simple.

3. Two matrices $[A_{ij}]$ and $[B_{ij}]$ can be found such that $[A_{ij}][B_{ij}] = [nn_{ij} - mm_{ij}]$ and such that $[\bar{y}]_{13} = [A_{ij}][R]_1^{-1}/n$ and $[\bar{y}]_{21} = [R]_2^{-1}[B_{ij}]/n$ are satisfactory submatrices.

Case B

1. $[n_{ij}]/m$ is realizable as a lossless N-port network and further,

the residue matrix is positive definite at each pole.

2. n/m has only j-axis poles and those are simple.

3. Two matrices $[A_{ij}]$ and $[B_{ij}]$ can be found such that $[A_{ij}][B_{ij}] = [mm_{ij} - nn_{ij}]$ and such that $[\bar{y}]_{13} = [A_{ij}][R]_1^{-1}/m$ and $[\bar{y}]_{21} = [R]_2^{-1}[B_{ij}]/m$ are satisfactory submatrices.

Satisfactory or acceptable submatrices for $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ are $N \times N$ matrices whose elements are real rational odd functions having only simple j-axis poles (this implies only simple j-axis poles with real residues) and, furthermore, any pole of any element of $[\bar{y}]_{13}$ or $[\bar{y}]_{21}$ is a pole of m_{ii}/n for Case A or n_{ii}/m for Case B for $i = 1,2,\ldots,N$. The matrices $[R]_1$ and $[R]_2$ are $N \times N$ diagonal matrices with nonzero, constant elements. As such, they have no influence on whether $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ are satisfactory or not.

Proof

Since the proof for Case A is identical to that for Case B, no distinction will be made between the two cases. Conditions 1, 2, 3 and the equations for the synthesis procedures ensure that the elements of the submatrices $[\bar{y}]_{11}$, $[\bar{y}]_{13}$, $[\bar{y}]_{21}$ and $[\bar{y}]_{23}$ are real rational functions that have only simple j-axis poles with real residues. Since $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ can be chosen freely, they can be chosen to be diagonal matrices with the diagonal elements $\bar{y}_{ii}$ of the matrix $[\bar{y}]$ satisfying the following conditions for $i = N + 1, N + 2,\ldots,3N$:

1. Any pole of any element of $[\bar{y}]_{11}$ or $[\bar{y}]_{23}$ is a pole of every $\bar{y}_{ii}$.

2. $\bar{y}_{ii}$ is a reactance function.

From the above discussion it can be seen that Condition 2 of the theorem of Appendix I is satisfied for the $3N \times 3N$ matrix $[\bar{y}]$.

Condition 1 of Theorem 3 and the above choice of $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ ensure that Condition 1 of the theorem of Appendix I is satisfied for $[\bar{y}]$.

The residue matrix in each pole must now be considered. First, consider any pole s_v of any element of $[\bar{y}]_{11}$, $[\bar{y}]_{13}$ or $[\bar{y}]_{21}$. Conditions 1 and 3 of Theorem 3 ensure that s_v is a pole of $\bar{y}_{ii}$ for $i = 1,2,\ldots,N$. Because of the choice of $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ which was described above, s_v is also a pole of $\bar{y}_{ii}$ for $i = N+1, N+2,\ldots,3N$. Hence, s_v is a pole of every $\bar{y}_{ii}$ for $i = 1, 2,\ldots,3N$. Let the residue matrix for the pole at s_v be $[k_{ij}]$ for $i,j = 1,2,\ldots,3N$. The positive definite requirement of Condition 1 of Theorem 3 ensures that (8)

$$k_{11} > 0$$

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2$$

$$\cdots\cdots\cdots\cdots$$

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2,\ldots,N\,.$$

Clearly, the residue $k_{N+1,N+1}$ can be chosen large enough such that

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2,\ldots,N+1\,.$$

Then, $k_{N+2,N+2}$ can be chosen large enough such that

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2,\ldots,N+2\,.$$

This process can be continued such that

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2,\ldots,N+3\,.$$

$$\cdots\cdots\cdots\cdots$$

$$|k_{ij}| > 0 \quad \text{for } i,j = 1,2,\ldots,3N\,.$$

The above sequence of inequalities is sufficient to ensure that the matrix of residues at the pole s_v is positive definite. Thus, for the pole at s_v, Condition 3 of the theorem of Appendix I is satisfied.

Next, any pole s_u of any element of $[\bar{y}]_{23}$ will be considered. By the choice of $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$ described previously, s_u is also a pole of every $\bar{y}_{ii}$ for $i = N + 1, N + 2, \ldots, 3N$. If s_u is also a pole of every $\bar{y}_{ii}$ for $i = 1, 2, \ldots, N$, then it falls into the same class as s_v and is covered by the discussion of the preceding paragraph. However, if s_u is not a pole of every $\bar{y}_{ii}$ for $i = 1, 2, \ldots, N$, then Condition 1 of Theorem 3 ensures that s_u is not a pole of any of the functions of $[\bar{y}]_{11}$. Furthermore, Condition 3 ensures that s_u is not a pole of any of the functions of $[\bar{y}]_{13}$ or $[\bar{y}]_{21}$. Hence $[k]_{11} = [k]_{13} = [k]_{21} = [0]$ for the pole s_u. Since the first N rows (or columns) of the residue matrix are zero, its rank cannot exceed $2N$. Clearly, the residues $k_{N+1,N+1}, \ldots, k_{3N,3N}$ can be chosen such that

$$k_{N+1,N+1} > 0$$

$$|k_{ij}| > 0 \quad \text{for } i,j = N+1, N+2$$

$$|k_{ij}| > 0 \quad \text{for } i,j = N+1, N+2, N+3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|k_{ij}| > 0 \quad \text{for } i,j = N+1, N+2, \ldots, 3N \, .$$

Thus, the rank of the matrix is $2N$ and the $2N$ principal minors in the above inequalities are all positive. This is sufficient to ensure that the residue matrix for the pole at s_u is positive semidefinite (8).

The poles, s_v and s_u, considered in the above paragraphs include all possible poles of $[\bar{y}]$ that the conditions of Theorem 3 allow. Therefore the residue matrix at each pole of $[\bar{y}]$ is positive semidefinite; i.e., Condition 3 of the Theorem of Appendix I is satisfied. It has already been shown that Conditions 1 and 2 of that theorem are satisfied. Therefore, the conditions listed under Case A or Case B of Theorem 3 are sufficient to ensure that the matrix $[\bar{y}]$ is realizable as a lossless 3N-port network and, thus, to assure the success of the synthesis procedure. This completes the proof of Theorem 3.

Before proceeding to the problem of matrix factorization, some conditions will be listed that are necessary for the existence of satisfactory submatrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$. For Case A these are:

1. If m_{ij} and n have a common factor F, then F^2 is a factor of $nn_{ij} - mm_{ij}$.

2. If m_{ij}/n has a zero at infinity, $\deg(nn_{ij} - mm_{ij}) \leq 2 \deg n - 2$. Otherwise, $\deg(nn_{ij} - mm_{ij}) \leq 2 \deg n + 2$.

3. If m_{ij}/n has a zero at the origin, then $(nn_{ij} - mm_{ij})/n^2$ has a zero there (second or higher order). For all three conditions $i,j = 1,2,\ldots,N$. The necessity of these conditions can be established by assuming that $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ are satisfactory submatrices and then using the necessary and sufficient conditions for lossless networks along with the equations for the synthesis procedure. In Condition 1, if F is a factor of both m_{ij} and n, then F^2 must be a factor of every element of $[nn_{ij} - mm_{ij}]$. Otherwise, F would contribute a pole to either $[\bar{y}]_{13}$ or $[\bar{y}]_{21}$ making it unsatisfactory.

For Condition 2, if $\bar{y}_{ii}$ has a zero at infinity, every element of

$[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ must have a zero there. Thus, the degree of any element of $[A_{ij}]$ or $[B_{ij}]$ cannot exceed deg n - 1. Therefore, deg $(nn_{ij} - mm_{ij}) \leq 2$ deg n - 2. If $\bar{y}_{ii}$ has a pole at infinity, it must be simple. Likewise, possible poles of the elements of $[\bar{y}]_{13}$ or $[\bar{y}]_{21}$ must be simple. Therefore deg $(nn_{ij} - mm_{ij}) \leq 2$ deg n + 2.

For Condition 3, if $\bar{y}_{ii}$ has a zero at the origin, every element of $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ must have a zero there. Thus, every element of $[nn_{ij} - mm_{ij}]/n^2$ must have at least a second order zero there.

A corresponding list of conditions can be obtained for Case B by interchanging m_{ij} with n_{ij} and m with n. If the appropriate list of conditions (Case A or Case B) is not satisfied for a prescribed matrix, there is no hope of finding satisfactory submatrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$, and the synthesis procedure for that particular prescribed admittance matrix should be abandoned. If none of the conditions are violated, then finding $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ becomes a problem of factoring a matrix of polynomial elements. Most 2 x 2 matrices of polynomials can be factored.

Matrix Factorization Techniques

In the matrix factorization required for the synthesis procedure, the elements are even, so, s^2 may be replaced by x giving a matrix of general polynomial elements. In this section, techniques for factoring a 2 x 2 matrix of polynomials will be considered. An existing technique and its sufficient conditions for factoring an N x N matrix of polynomials are given in Appendix II. The class of matrices that can be factored by the techniques given there is severely limited. The need to enlarge the class of matrices that could be factored motivated the development of a new technique for factoring 2×2 matrices. This new technique, which makes possible the factorization of a larger class of matrices, is presented in this section. Throughout this section, the indices take on the values $i,j = 1,2$.

Consider a 2×2 matrix $[r_{ij}]$ whose elements are polynomials of degree L. If the matrix factorization technique of Appendix II is to be used, reference to Equation (A5) of that appendix shows that in order to find two matrices $[p_{ij}]$ and $[q_{ij}]$ such that $[p_{ij}][q_{ij}] = [r_{ij}]$ with deg $p_{ij} = L - k$ and deg $q_{ij} = k$, it is sufficient that det r_{ij} have at least $L + k$ distinct real zeros. However, only $2k$ of those zeros are actually used in the procedure of finding the matrix factors. Usually, if det r_{ij} has $2k$ distinct real zeros, the factorization can be performed. If det r_{ij} has less than $2k$ real zeros, the desired factorization cannot be completely accomplished by the procedure of Appendix II unless complex coefficients are allowed. Clearly, complex coefficients in the submatrices of the $3N \times 3N$ admittance matrix $[\bar{y}]$ would render that matrix unrealizable as a lossless network. This being true, the technique of Appendix II is useful only when performed with real zeros of the determinant. Because of its simplicity, this method should be used until the desired factorization is accomplished or until it is no longer applicable. After exhausting this procedure, if det r_{ij} has complex zeros, the technique developed below should be used.

To develop the technique for finding a matrix factor whose determinant has a pair of conjugate complex zeros suppose that

1. $[r_{ij}] = \begin{bmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{bmatrix}$ is the matrix to be factored.

2. The degree of r_{ij} is L.

3. $|r_{ij}| \neq 0$.

4. x_1 and x_1^* are conjugate complex zeros of $|r_{ij}|$ and Im $x_1 \neq 0$.

5. $r_{12}(x_1)$ and $r_{22}(x_1)$ are not both zero.

6. $\text{Im}\,\dfrac{r_{11}(x_1)}{r_{12}(x_1)} \neq 0$ if $r_{12}(x_1) \neq 0$ or $\text{Im}\,\dfrac{r_{21}(x_1)}{r_{22}(x_1)} \neq 0$ if $r_{22}(x_1) \neq 0$.

Let

$$[H_{ij}] = \begin{bmatrix} H_{11}(x) & 0 \\ H_{21}(x) & 1 \end{bmatrix}$$

$$\begin{aligned} H_{11}(x) &= a_{11}x - b_{11} \\ H_{21}(x) &= a_{21}x - b_{21} \end{aligned} \tag{44}$$

where a_{11}, a_{21}, b_{11}, b_{21} are real constants. Forming the matrix product $[r_{ij}][H_{ij}]$ gives

$$[r_{ij}][H_{ij}] = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ H_{21} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}H_{11} + r_{12}H_{21} & r_{12} \\ r_{21}H_{11} + r_{22}H_{21} & r_{22} \end{bmatrix} \tag{45}$$

The procedure is to determine polynomials H_{11} and H_{21} such that both elements of the first column of the right-hand member of Equation (45) contain the zeros x_1 and x_1^*. To accomplish this set

$$\begin{aligned} r_{11}(x_1)H_{11}(x_1) + r_{12}(x_1)H_{21}(x_1) &= 0 \\ r_{21}(x_1)H_{11}(x_1) + r_{22}(x_1)H_{21}(x_1) &= 0 \end{aligned} \tag{46}$$

Using Equations (44), Equations (46) become

$$\begin{aligned} x_1r_{11}(x_1)a_{11} - r_{11}(x_1)b_{11} + x_1r_{12}(x_1)a_{21} - r_{12}(x_1)b_{21} &= 0 \\ x_1r_{21}(x_1)a_{11} - r_{21}(x_1)b_{11} + x_1r_{22}(x_1)a_{21} - r_{22}(x_1)b_{21} &= 0 \end{aligned} \tag{47}$$

Equations (47) form a system of homogeneous linear equations with unknowns a_{11}, b_{11}, a_{21} and b_{21} and with the coefficient matrix

$$\begin{bmatrix} x_1r_{11}(x_1) & -r_{11}(x_1) & x_1r_{12}(x_1) & -r_{12}(x_1) \\ x_1r_{21}(x_1) & -r_{21}(x_1) & x_1r_{22}(x_1) & -r_{22}(x_1) \end{bmatrix}.$$

Recalling that $|r_{ij}(x_1)| = 0$, it is easily seen that the rank of this coefficient matrix is one. This means that the system of equations has a nontrivial solution and further, that three of the unknowns can be chosen arbitrarily. After choosing the first two unknowns to be real, the third is chosen to be real and to have a value such that the fourth is real. Following this procedure, choose $a_{11} = 1$, choose b_{11} to be any real number such that b_{11} is not a zero of det r_{ij}.

The unknown a_{21} will be chosen after Equations (47) are solved for b_{21}. The hypothesis requires that either $r_{12}(x_1)$ or $r_{22}(x_1)$ (or both) be nonzero. Say $r_{22}(x_1) \neq 0$. Then, the second of Equations (47) gives

$$b_{21} = \frac{r_{21}(x_1)}{r_{22}(x_1)} (x_1 - b_{11}) + x_1 a_{21} \tag{48}$$

Since the hypothesis excludes the case of $\text{Im } x_1 = 0$, it is clear that a real number can be chosen for a_{21} such that the imaginary part of the right-hand member of Equation (48) is zero. This requires that

$$a_{21} = - \frac{\text{Im}\left\{\dfrac{r_{21}(x_1)}{r_{22}(x_1)} (x_1 - b_{11})\right\}}{\text{Im } x_1} \tag{49}$$

With this choice of a_{21}, b_{21} is given by Equation (48). This completes the choice of a_{11}, b_{11}, a_{21} and b_{21} and thus, polynomials H_{11} and H_{21} have been determined such that x_1 is a solution to the equations

$$\begin{aligned} r_{11}H_{11} + r_{12}H_{21} &= 0 \\ r_{21}H_{11} + r_{22}H_{21} &= 0 \end{aligned} \tag{50}$$

But, the left-hand members of Equations (50) are polynomials in x with real coefficients and therefore, x_1^* is also a solution to Equations (50). Hence, both elements of the first column of the right-hand member of Equation (45) contain the factors $(x - x_1)$ and $(x - x_1^*)$. Equation (45) can now be written as

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & r_{12} \\ g_{21} & r_{22} \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \tag{51}$$

where

$$G = (x - x_1)(x - x_1^*) = x^2 - 2\ \mathrm{Re}\ x_1 x + (\mathrm{Re}\ x_1)^2 + (\mathrm{Im}\ x_1)^2$$

$$\begin{aligned} g_{11} &= \frac{r_{11}H_{11} + r_{12}H_{21}}{G} \\ g_{21} &= \frac{r_{21}H_{11} + r_{22}H_{21}}{G} \end{aligned} \tag{52}$$

The elements g_{11} and g_{21} are polynomials of degree $L - 1$.

Now, consider the matrix $\begin{bmatrix} g_{11} & r_{12} \\ g_{21} & r_{22} \end{bmatrix}$.

The determinant of this matrix has a distinct real zero at $x = b_{11}$. The factor $(x - b_{11})$ will be removed from the second column of this matrix by means of the procedure of Appendix II. To do this the following matrix product is formed:

$$\begin{bmatrix} g_{11} & r_{12} \\ g_{21} & r_{22} \end{bmatrix}\begin{bmatrix} 1 & c_{12} \\ 0 & c_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & c_{12}g_{11} + c_{22}r_{12} \\ g_{21} & c_{12}g_{21} + c_{22}r_{22} \end{bmatrix} \tag{53}$$

where c_{12} and c_{22} are real constants. Forcing both elements of the second column of the right-hand member of Equation (53) to have zeros at $x = b_{11}$ gives

$$c_{12}g_{11}(b_{11}) + c_{22}r_{12}(b_{11}) = 0$$
$$c_{12}g_{21}(b_{11}) + c_{22}r_{22}(b_{11}) = 0 \tag{54}$$

This is a system of homogeneous linear equations with unknowns c_{12} and c_{22}. Since the determinant of the coefficient matrix if zero, the system of Equations (54) has a nontrivial solution. The constant c_{22} can be chosen nonzero if either $g_{11}(b_{11}) \neq 0$ or $g_{21}(b_{11}) \neq 0$. Before showing that both $g_{11}(b_{11})$ and $g_{21}(b_{11})$ are not zero, it must be shown that $H_{21}(b_{11}) \neq 0$. Substituting Equation (48) into the second of Equations (44) and evaluating at $x = b_{11}$ give

$$\begin{aligned} H_{21}(b_{11}) &= a_{21}b_{11} - \frac{r_{21}(x_1)}{r_{22}(x_1)}(x_1 - b_{11}) - x_1 a_{21} \\ &= -\left(\frac{r_{21}(x_1)}{r_{22}(x_1)} + a_{21}\right)(x_1 - b_{11}) \end{aligned} \tag{55}$$

From the hypothesis, $\operatorname{Im} \frac{r_{21}(x_1)}{r_{22}(x_1)} \neq 0$ and $\operatorname{Im} x_1 \neq 0$. However, the constants a_{21} and b_{11} are real. Therefore, neither factor of the right-hand member of Equation (55) can be zero. Thus $H_{21}(b_{11}) \neq 0$.

Returning now to Equations (52) and evaluating at $x = b_{11}$ give

$$g_{11}(b_{11}) = \frac{r_{11}(b_{11})H_{11}(b_{11}) + r_{12}(b_{11})H_{21}(b_{11})}{G(b_{11})}$$
$$g_{21}(b_{11}) = \frac{r_{21}(b_{11})H_{11}(b_{11}) + r_{22}(b_{11})H_{21}(b_{11})}{G(b_{11})} \tag{56}$$

But, $H_{11}(b_{11}) = 0$ and as was shown above $H_{21}(b_{11}) \neq 0$. Furthermore,

$G(b_{11}) \neq 0$ because G has only complex zeros while b_{11} is real. So, Equation (56) become

$$g_{11}(b_{11}) = \frac{r_{12}(b_{11})H_{21}(b_{11})}{G(b_{11})}$$
$$g_{21}(b_{11}) = \frac{r_{22}(b_{11})H_{21}(b_{11})}{G(b_{11})} \tag{57}$$

Both $r_{12}(b_{11})$ and $r_{22}(b_{11})$ cannot be zero. Otherwise, b_{11} would be a zero of det r_{ij}, but this is contrary to the choice of b_{11}. Therefore, both $g_{11}(b_{11})$ and $g_{21}(b_{11})$ cannot be zero.

Now, choosing $c_{22} = 1$ and using the first and second of Equations (54) give, respectively,

$$c_{12} = -\frac{r_{12}(b_{11})}{g_{11}(b_{11})} \quad \text{if} \quad g_{11}(b_{11}) \neq 0$$
$$c_{12} = -\frac{r_{22}(b_{11})}{g_{21}(b_{11})} \quad \text{if} \quad g_{21}(b_{11}) \neq 0 \tag{58}$$

With the above values of c_{12} and c_{22}, both elements of the second column of the right-hand member of Equation (53) contains the factor $(x - b_{11})$. Thus, Equation (53) may be written as

$$\begin{bmatrix} g_{11} & r_{12} \\ g_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 1 & c_{12} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_{11} \end{bmatrix} \tag{59}$$

where

$$g_{12} = \frac{c_{12}g_{11} + r_{12}}{H_{11}}$$

$$g_{22} = \frac{c_{12}g_{21} + r_{22}}{H_{11}} \tag{60}$$

The elements g_{12} and g_{22} are polynomials of degree $L - 1$. Equation (59) can be rewritten as

$$\begin{bmatrix} g_{11} & r_{12} \\ g_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_{11} \end{bmatrix} \begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \tag{61}$$

Substituting Equation (61) into Equation (51) gives

$$\begin{aligned} [r_{ij}][H_{ij}] &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H_{11} \end{bmatrix} \begin{bmatrix} 1 & -c_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} G & -c_{12} \\ 0 & H_{11} \end{bmatrix} \end{aligned} \tag{62}$$

Now, consider the matrix $\begin{bmatrix} G & -c_{12} \\ 0 & H_{11} \end{bmatrix}$.

The determinant of this matrix has a zero at $x = b_{11}$. The factor $(x - b_{11})$ will be removed from the first column of this matrix by the procedure of Appendix II. To do this the matrix product is formed as indicated below and both elements of the first column of that product are required to have a zero at $x = b_{11}$.

$$\begin{bmatrix} G & -c_{12} \\ 0 & H_{11} \end{bmatrix}\begin{bmatrix} d_{11} & 0 \\ d_{21} & 1 \end{bmatrix} = \begin{bmatrix} d_{11}G - d_{21}c_{12} & -c_{12} \\ d_{21}H_{11} & H_{11} \end{bmatrix} \tag{63}$$

where d_{11} and d_{21} are real constants. Hence, it is necessary that

$$\begin{aligned} d_{11}G(b_{11}) - d_{21}c_{12} &= 0 \\ d_{21}H_{11}(b_{11}) &= 0 \end{aligned} \tag{64}$$

Since $H_{11}(b_{11}) = 0$, the second of Equations (64) is satisfied for any finite value of d_{21}. To ensure that there is a finite value of d_{21} and a nonzero value of d_{11} that satisfy the first of Equations (64), it must be shown that $c_{12} \neq 0$. To this end, Equations (57) are substituted into Equations (58). This gives

$$\begin{aligned} c_{12} &= - \frac{r_{12}(b_{11})G(b_{11})}{r_{12}(b_{11})H_{21}(b_{11})} = - \frac{G(b_{11})}{H_{21}(b_{11})} \\ c_{12} &= - \frac{r_{22}(b_{11})G(b_{11})}{r_{22}(b_{11})H_{21}(b_{11})} = - \frac{G(b_{11})}{H_{21}(b_{11})} \end{aligned} \tag{65}$$

Since G has only complex zeros and b_{11} is real, $G(b_{11})$ cannot be zero. Furthermore, since H_{21} has finite coefficients, $H_{21}(b_{11})$ is finite. Therefore, from either of Equations (65), $c_{12} \neq 0$.

Now, choosing $d_{11} = 1$ in the first of Equations (64) and solving for d_{21} give

$$d_{21} = \frac{G(b_{11})}{c_{12}}$$

Using Equations (65) this becomes

$$d_{21} = -H_{21}(b_{11}) \tag{66}$$

Making use of Equations (65) and (66), Equation (63) can be written as

$$\begin{bmatrix} G & -c_{12} \\ 0 & H_{11} \end{bmatrix} = \begin{bmatrix} G - G(b_{11}) & -c_{12} \\ -H_{21}(b_{11})H_{11} & H_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ H_{21}(b_{11}) & 1 \end{bmatrix} \tag{67}$$

Since both elements of the first column of the first factor of the right-hand member of Equation (67) are zero at $x = b_{11}$, that equation can be written as

$$\begin{aligned} \begin{bmatrix} G & -c_{12} \\ 0 & H_{11} \end{bmatrix} &= \begin{bmatrix} h_{11} & -c_{12} \\ -H_{21}(b_{11}) & H_{11} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ H_{21}(b_{11}) & 1 \end{bmatrix} \\ &= \begin{bmatrix} h_{11} & -c_{12} \\ -H_{21}(b_{11}) & H_{11} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ H_{21}(b_{11}) & 1 \end{bmatrix} \end{aligned} \tag{68}$$

where

$$h_{11} = \frac{G - G(b_{11})}{H_{11}} \tag{69}$$

is a first degree polynomial

Substituting Equation (68) into Equation (62) gives

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} h_{11} & -c_{12} \\ -H_{21}(b_{11}) & H_{11} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ H_{21}(b_{11}) & 1 \end{bmatrix} \tag{70}$$

If the final factor of Equation (70) can be made equal to $[H_{ij}]$, that equation can be postmultiplied by $[H_{ij}]^{-1}$ and the given matrix $[r_{ij}]$ will have been factored. To accomplish this, another constant ρ_{21} is introduced into Equation (70) as shown below.

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} h_{11} & -c_{12} \\ -H_{21}(b_{11}) & H_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\rho_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho_{21} & 1 \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ H_{21}(b_{11}) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} h_{11} + \rho_{21}c_{12} & -c_{12} \\ -H_{21}(b_{11}) - \rho_{21}H_{11} & H_{11} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ \rho_{21}H_{11} + H_{21}(b_{11}) & 1 \end{bmatrix} \quad (71)$$

The constant ρ_{21} must be determined such that

$$\rho_{21}H_{11} + H_{21}(b_{11}) = H_{21} \quad (72)$$

Using Equations (44) and the previous choice of $a_{11} = 1$, Equation (72) becomes

$$\rho_{21}(x - b_{11}) + a_{21}b_{11} - b_{21} = a_{21}x - b_{21}$$

Equating the corresponding coefficients of x gives

$$\rho_{21} = a_{21}$$

and

$$-\rho_{21}b_{11} + a_{21}b_{11} = 0$$

Both of these equations are satisfied by $\rho_{21} = a_{21}$. Therefore, Equation

(72) is satisfied when $\rho_{21} = a_{21}$. Under this condition Equation (71) becomes

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} h_{11} + a_{21}c_{12} & -c_{12} \\ -H_{21} & H_{11} \end{bmatrix} \begin{bmatrix} H_{11} & 0 \\ H_{21} & 1 \end{bmatrix}$$

Postmultiplying both sides by $[H_{ij}]^{-1}$ gives

$$[r_{ij}] = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} h_{11} + a_{21}c_{12} & -c_{12} \\ -H_{21} & H_{11} \end{bmatrix} \tag{73}$$

All symbols of Equation (73) have been previously defined. Reference to those definitions show that the elements of the first factor are polynomials of degree $L - 1$ while those of the second factor are either first degree polynomials or a constant.

Before closing this section a few remarks will be made regarding Conditions 5 and 6 of the hypothesis which is given at the beginning of this development. If both $r_{12}(x_1)$ and $r_{22}(x_1)$ are zero, then $r_{12}(x_1^*)$ and $r_{22}(x_1^*)$ are also zero and hence, the elements of the second column contain a common quadratic factor. This quadratic factor may be removed as the second element of a diagonal matrix. In some cases this may result in acceptable factors for the matrix $[r_{ij}]$. If Condition 5 is satisfied and Condition 6 fails, the polynomials H_{11} and H_{21} can be replaced by real constants. Say $r_{22}(x_1) \neq 0$ and $\operatorname{Im} \dfrac{r_{21}(x_1)}{r_{22}(x_1)} = 0$. Choosing $H_{11} = 1$, the first of Equations (46) gives $H_{21} = -\dfrac{r_{21}(x_1)}{r_{22}(x_1)}$, a real constant. The quadratic factor can then be removed from the first column of the product $[r_{ij}][H_{ij}]$ as the first element of a diagonal matrix. Postmultiplication by $[H_{ij}]^{-1}$ would complete the removal of the quadratic factor from the first column of $[r_{ij}]$. Again, this may or may not be an acceptable factorization.

In this section, a technique has been developed for factoring a given matrix having polynomial elements of degree L. One of the matrix factors has polynomial elements of degree at most L - 1 while the other has elements of degree one or less. This technique does not require that the determinant of the given matrix have any real zeros. Instead, it makes use of a pair of conjugate complex zeros. Certain conditions, which are given in the hypothesis at the beginning of the development, assure the success of the technique. However, these conditions are not very demanding and they impose practically no limitation on the procedure. If $[g_{ij}]$ of Equation (73) meets these conditions, the technique can be applied to that matrix. Most matrices will allow repeated application as long as the determinant has a pair of conjugate complex zeros. By means of the technique developed here along with that of Appendix II, it is possible to satisfactorily factor most matrices. The technique of this section is required in the example of the following section.

An Example

To illustrate the two-port synthesis procedure, a network will be found having the prescribed admittance matrix

$$\begin{aligned}[\tilde{y}] &= \frac{1}{m+n}[m_{ij} + n_{ij}] \\ &= \frac{1}{s^2+s+1}\begin{bmatrix} s^3 + 2s^2 + 9s + 8 & s^2 - 2 \\ s^3 + s^2 - 2 & 2s^3 + 3s^2 + 3s + 3 \end{bmatrix}\end{aligned} \tag{74}$$

In this example, Case A is indicated since $[m_{ij}]/n$ is realizable as a lossless network whereas, $[n_{ij}]/m$ is not realizable as such for $i,j = 1,2$. As the first step, Equation (33) gives

$$[\bar{y}]_{11} = \frac{[m_{ij}]}{n} = \frac{1}{s}\begin{bmatrix} 2s^2+8 & s^2-2 \\ s^2-2 & 3s^2+3 \end{bmatrix} \tag{75}$$

The next step of the procedure is to find satisfactory matrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ such that Equation (38) is satisfied. The numerator of that equation is

$$[nn_{ij} - mm_{ij}] = \begin{bmatrix} s(s^3+9) - (s^2+1)(2s^2+8) & s(0) - (s^2+1)(s^2-2) \\ s(s^3) - (s^2+1)(s^2-2) & s(2s^3+3s) - (s^2+1)(3s^2+3) \end{bmatrix}$$

$$= \begin{bmatrix} -s^4 - s^2 - 8 & -s^4 + s^2 + 2 \\ s^2 + 2 & -s^4 - 3s^2 - 3 \end{bmatrix} \tag{76}$$

Equation (38) becomes

$$[\bar{y}]_{13}[R]_1[R]_2[\bar{y}]_{21} = \frac{1}{s^2}\begin{bmatrix} -s^4 - s^2 - 8 & -s^4 + s^2 + 2 \\ s^2 + 2 & -s^4 - 3s^2 - 3 \end{bmatrix} \tag{77}$$

In order to identify acceptable submatrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ satisfying Equation (77), the matrix of the right-hand member of that equation must be factored into two matrices having even polynomial elements of degree two or less. Replacing s^2 by x, the matrix to be factored will be designated as $[r_{ij}]$. Thus,

$$[r_{ij}] = \begin{bmatrix} -x^2 - x - 8 & -x^2 + x + 2 \\ x + 2 & -x^2 - 3x - 3 \end{bmatrix} \tag{78}$$

Computing the determinant and its factors gives

$$\begin{aligned} |r_{ij}| &= x^4 + 5x^3 + 15x^2 + 23x + 20 \\ &= (x^2 + 2x + 5)(x^2 + 3x + 4) \end{aligned}$$

The first of these factors represents a pair of conjugate complex zeros at $x_1 = -1 + j2$ and $x_1^* = -1 - j2$ while the second represents a conjugate pair at x_2, $x_2^* = -\frac{3}{2} \pm j\frac{\sqrt{7}}{2}$. Since there are no real zeros, the factorization technique of Appendix II cannot be used; so, that developed in this chapter will be applied. The zeros x_1, x_1^* will be used to accomplish the removal of a matrix of linear elements. The constant b_{11} can be chosen to be any real number such that b_{11} is not a zero of det r_{ij}. Choosing $b_{11} = -1$ and using Equation (49) and then Equation (48) give

$$a_{21} = -\frac{\operatorname{Im}\left\{\dfrac{r_{21}(x_1)}{r_{22}(x_1)}\right\}(x_1 - b_{11})}{\operatorname{Im} x_1}$$

$$= -\frac{\operatorname{Im}\left\{\dfrac{1 + j2}{3 - j2}\right\}(j2)}{2} = \frac{1}{13}$$

$$b_{21} = \frac{r_{21}(x_1)}{r_{22}(x_1)}(x_1 - b_{11}) + x_1 a_{21}$$

$$= \frac{1 + j2}{3 - j2} (j2) + \frac{1}{13} (-1 + j2)$$

$$= - \frac{16}{13} - j \frac{2}{13} - \frac{1}{13} + j \frac{2}{13} = - \frac{17}{13}$$

Equations (44) give

$$\begin{aligned} H_{11} &= x + 1 \\ H_{21} &= \frac{1}{13} x + \frac{17}{13} \end{aligned} \tag{79}$$

It is not necessary to repeat all of the intermediate steps of the general development of the factorization technique. Reference to the final result, Equation (73), reveals the functions and constants that must be computed.

From Equations (52) and (79)

$$g_{11} = \frac{r_{11}H_{11} + r_{12}H_{21}}{G}$$

$$= \frac{(-x^2 - x - 8)(x + 1) + (-x^2 + x + 2)(\frac{1}{13} x + \frac{17}{13})}{x^2 + 2x + 5}$$

$$= \frac{1}{13} (-14x - 14)$$

$$g_{21} = \frac{r_{21}H_{11} + r_{22}H_{21}}{G}$$

$$= \frac{(x + 2)(x + 1) + (-x^2 - 3x - 3)(\frac{1}{13} x + \frac{17}{13})}{x^2 + 2x + 5}$$

$$= \frac{1}{13} (-x - 5)$$

The constant c_{12} may be calculated from either Equations (58) or (65). Using Equations (65) gives

$$c_{12} = -\frac{G(b_{11})}{H_{21}(b_{11})}$$

$$= -\frac{(-1)^2 + 2(-1) + 5}{\frac{1}{13}(-1 + 17)} = -\frac{13}{4}$$

From Equations (60)

$$g_{12} = \frac{c_{12}g_{11} + r_{12}}{H_{11}}$$

$$= \frac{-\frac{13}{4}\frac{1}{13}(-14x - 14) + (-x^2 + x + 2)}{x + 1}$$

$$= \frac{1}{4}(-4x + 22)$$

$$g_{22} = \frac{c_{12}g_{21} + r_{22}}{H_{11}}$$

$$= \frac{-\frac{13}{4}\frac{1}{13}(-x - 5) + (-x^2 - 3x - 3)}{x + 1}$$

$$= \frac{1}{4}(-4x - 7)$$

From Equation (69)

$$h_{11} = \frac{G - G(b_{11})}{H_{11}}$$

$$= \frac{x^2 + 2x + 5 - 4}{x + 1}$$

$$= x + 1$$

The calculation of the constants and functions appearing in Equation (73) has been completed. Substituting these values into Equation (73) gives

$$[r_{ij}] = \begin{bmatrix} \frac{1}{13}(-14x - 14) & \frac{1}{4}(-4x + 22) \\ \\ \frac{1}{13}(-x - 5) & \frac{1}{4}(-4x - 7) \end{bmatrix} \begin{bmatrix} \frac{1}{4}(4x + 3) & \frac{13}{4} \\ \\ -\frac{1}{13}(x + 17) & x + 1 \end{bmatrix}$$

The above equation gives the factors of the matrix $[r_{ij}]$ as defined by Equation (78).

Replacing x by s^2, the factors of $[r_{ij}]$ become the factors of the matrix of the right-hand member of Equation (77). Hence, Equation (77) is satisfied by

$$[\overline{y}]_{13} = \frac{1}{s} \begin{bmatrix} \frac{1}{13}(-14s^2 - 14) & \frac{1}{4}(-4s^2 + 22) \\ \\ \frac{1}{13}(-s^2 - 5) & \frac{1}{4}(-4s^2 - 7) \end{bmatrix} [R]_1^{-1} \qquad (80)$$

and

$$[\overline{y}]_{21} = \frac{1}{s}[R]_2^{-1} \begin{bmatrix} \frac{1}{4}(4s^2 + 3) & \frac{13}{4} \\ \\ -\frac{1}{13}(s^2 + 17) & s^2 + 1 \end{bmatrix} \qquad (81)$$

The next step of the synthesis procedure is to choose the diagonal matrices $[R]_1$ and $[R]_2$. The choice of any nonzero values for the elements of these matrices is acceptable. Extremely small values will require extremely large residues in the poles of $\overline{y}_{ii}$ for $i = 3,4,5,6$; whereas, large values may allow the realization without transformers. If, at every pole of $[\bar{y}]_{11}$, the residue matrix satisfies the dominant condition with the inequality sign, then the elements of $[R]_1$ and $[R]_2$ can be chosen large enough so that the residue matrix of $[\bar{y}]$ is dominant at every pole. In this case no transformers are required. The following choices of $[R]_1$ and $[R]_2$ will allow transformerless realization in this example:

$$[R]_1 = \begin{bmatrix} \frac{40}{13} & 0 \\ 0 & \frac{40}{4} \end{bmatrix} \tag{82}$$

$$[R]_2 = \begin{bmatrix} \frac{40}{4} & 0 \\ 0 & \frac{40}{13} \end{bmatrix} \tag{83}$$

Using these matrices, Equations (80) and (81) become

$$[\bar{y}]_{13} = \frac{1}{40s} \begin{bmatrix} -14s^2 - 14 & -4s^2 + 22 \\ -s^2 - 5 & -4s^2 - 7 \end{bmatrix} \tag{84}$$

and

$$[\bar{y}]_{21} = \frac{1}{40s} \begin{bmatrix} 4s^2 + 3 & 13 \\ -s^2 + 17 & 13s^2 + 13 \end{bmatrix} \tag{85}$$

The coefficient matrix as determined by Equations (37), (82) and (83) is

$$[R] = \begin{bmatrix} \frac{1600}{52} & 0 \\ 0 & \frac{1600}{52} \end{bmatrix} \tag{86}$$

The final submatrix to be specified by the procedure is given by Equation (40) to be

$$[\bar{y}]_{23} = -\frac{s^2+1}{s}\begin{bmatrix} \frac{52}{1600} & 0 \\ 0 & \frac{52}{1600} \end{bmatrix} \tag{87}$$

There are two remaining submatrices, $[\bar{y}]_{22}$ and $[\bar{y}]_{33}$. The off diagonal terms of these will be set equal to zero and the functions $\bar{y}_{ii}$ for $i = 3,4,5,6$ need not be specified explicitly. Instead, their residues at each pole will be allowed to have the value that satisfies the dominant condition with the equal sign.

The final step is the realization of the lossless network and the connection of the controlled sources. Using the submatrices given by Equations (84), (85) and (87), the admittance matrix for the lossless six-port network is

$$[\bar{y}] = s\begin{bmatrix} 2 & 1 & 0.1 & -0.025 & -0.35 & -0.1 \\ 1 & 3 & 0 & 0.325 & -0.025 & -0.1 \\ 0.1 & 0 & k_{33}^{(\infty)} & 0 & -0.0325 & 0 \\ -0.025 & 0.325 & 0 & k_{44}^{(\infty)} & 0 & -0.0325 \\ -0.35 & -0.025 & -0.0325 & 0 & k_{55}^{(\infty)} & 0 \\ -0.1 & -0.1 & 0 & -0.0325 & 0 & k_{66}^{(\infty)} \end{bmatrix} + \frac{1}{s}\begin{bmatrix} 8 & -2 & 0.075 & -0.425 & -0.35 & 0.55 \\ -2 & 3 & 0.325 & 0.325 & -0.125 & -0.175 \\ 0.075 & 0.325 & k_{33}^{(0)} & 0 & -0.0325 & 0 \\ -0.425 & 0.325 & 0 & k_{44}^{(0)} & 0 & -0.0325 \\ -0.35 & -0.125 & -0.0325 & 0 & k_{55}^{(0)} & 0 \\ 0.55 & -0.175 & 0 & -0.0325 & 0 & k_{66}^{(0)} \end{bmatrix} \tag{88}$$

Networks realizing the first and second terms of Equation (88) are shown in Figures 4 and 5 respectively. These networks are connected in parallel to give the lossless-network realization of $[\bar{y}]$.

The coefficients of the controlled sources are determined by the matrix $[R]$. From Equation (86), $R_1 = R_2 = \frac{1600}{52}$. Connecting the current-controlled voltage sources as indicated in Figure 6 completes the realization for the admittance matrix prescribed by Equation (74).

Summary

In this chapter the main objective has been the development of a synthesis procedure and some sufficient conditions for a lossless two-port network with two current-controlled voltage sources embedded in it. The synthesis procedure derived is applicable to the lossless N-port network with N current-controlled voltage sources embedded in it provided an N x N matrix of polynomial elements can be satisfactorily factored. A new technique has been developed for factoring 2 x 2 matrices. An example illustrating the synthesis procedure was presented. This example required the use of the new factorization technique.

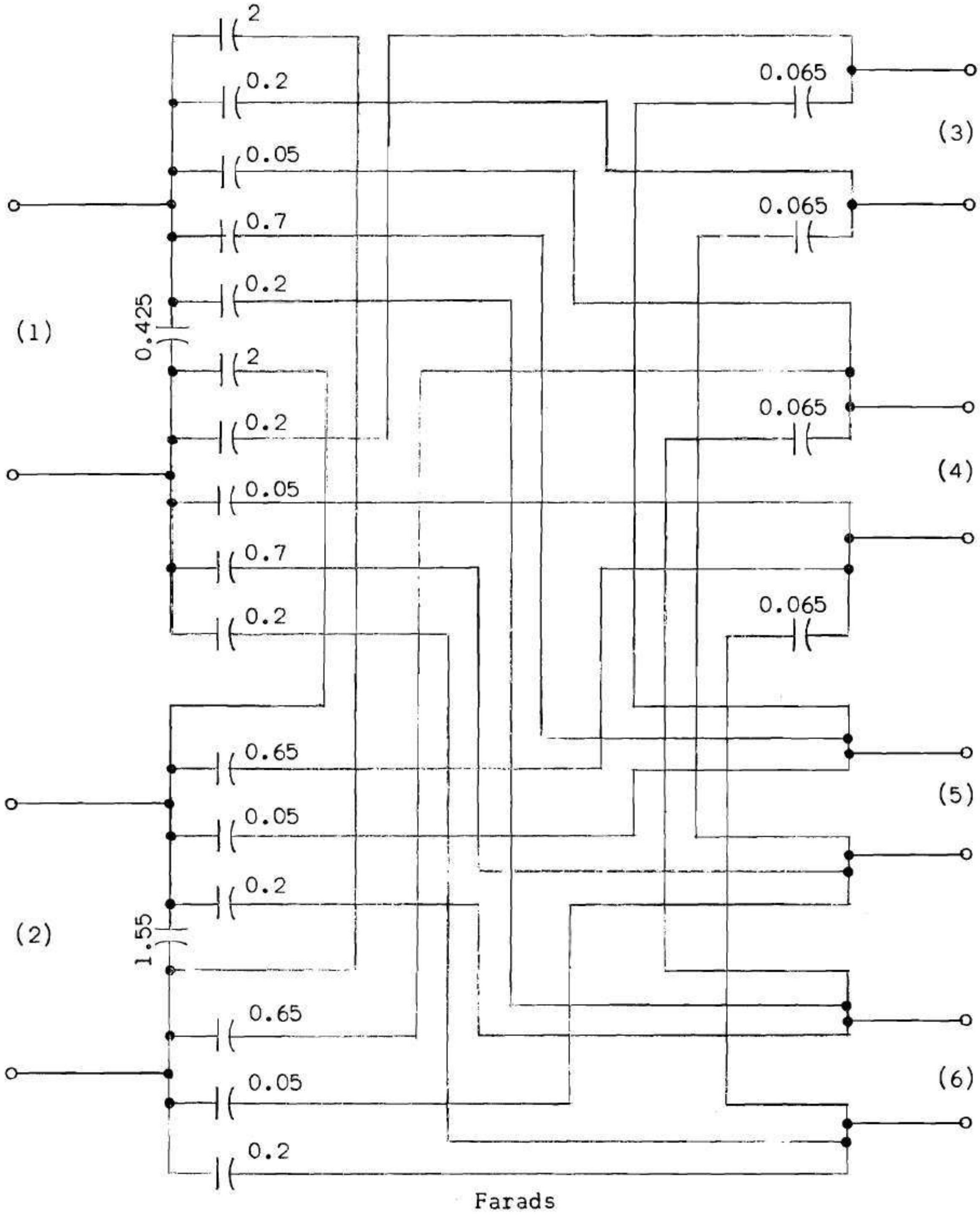


Figure 4. Network Realizing the First Term of Equations (88).

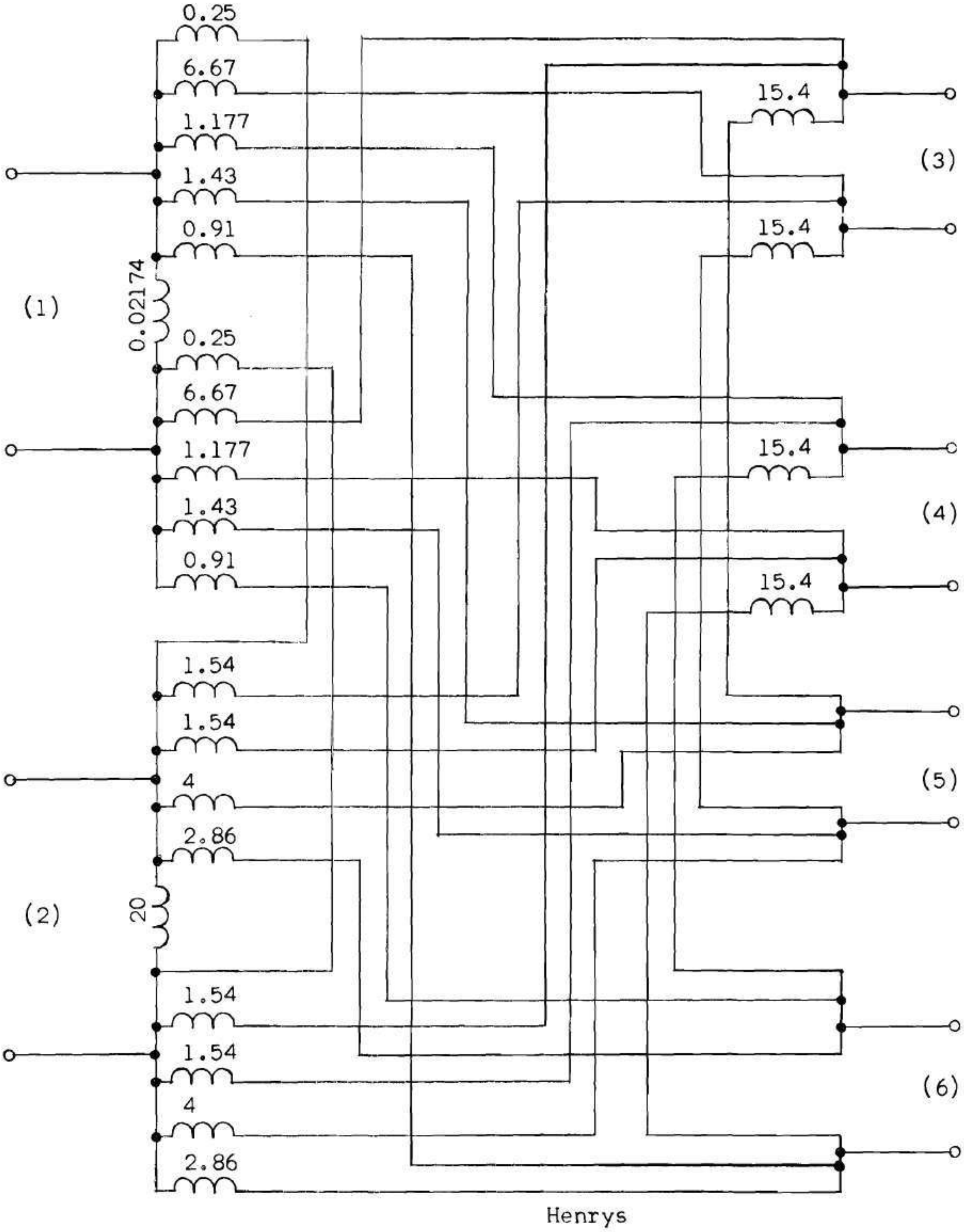


Figure 5. Network Realizing the Second Term of Equation (88).

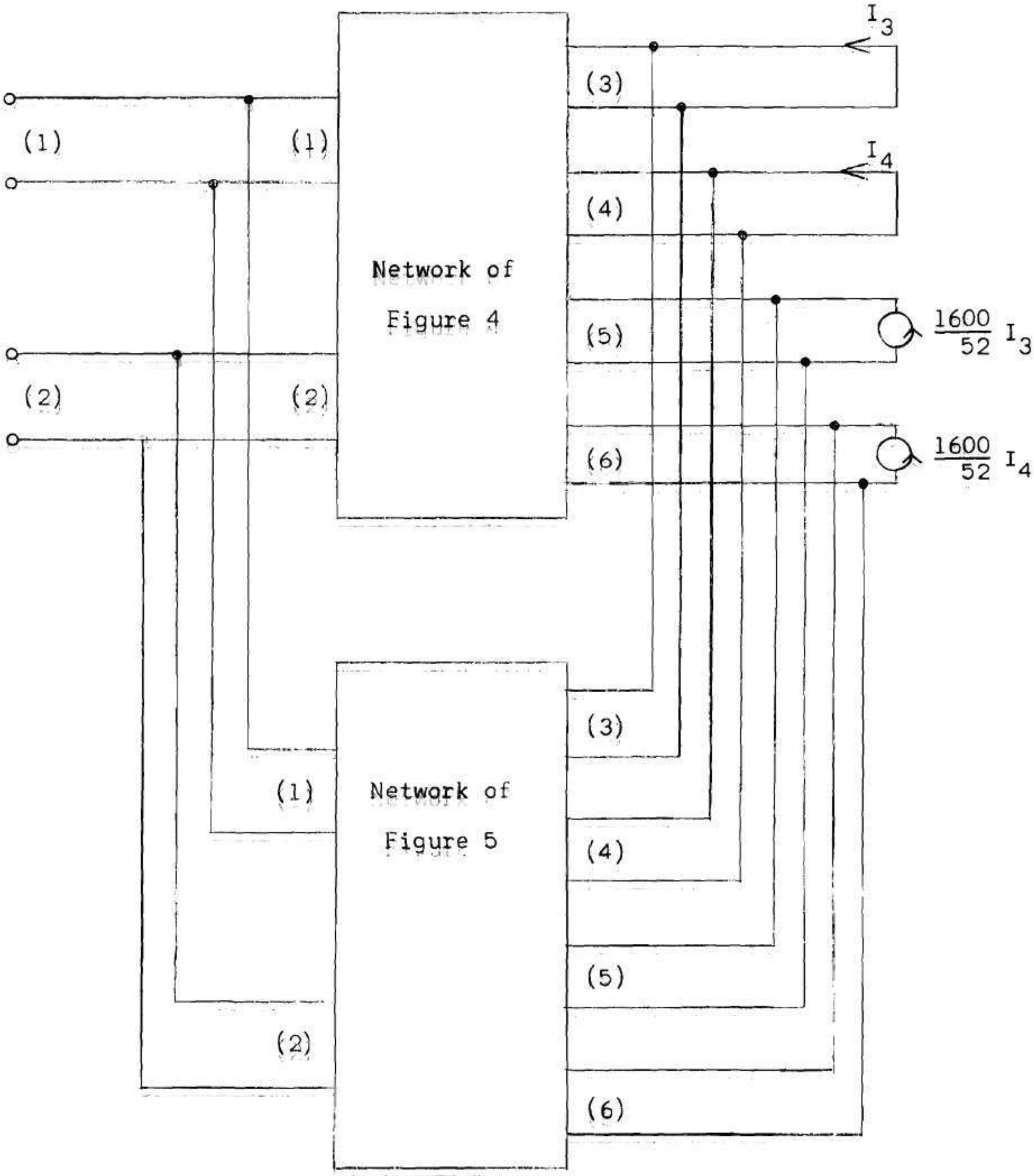


Figure 6. Network Realizing the Admittance Matrix of Equation (74).

CHAPTER IV

EXTENSIONS AND APPLICATIONS

The extension of the synthesis procedure to the N-port case will be considered in the first section of this chapter. As was noted in Chapter III, the synthesis procedure derived there is applicable to the N-port case provided an $N \times N$ matrix of polynomials can be factored. So, the only additional problem to be considered is that of factoring the $N \times N$ matrix. After consideration of this extension, the remainder of the chapter will be devoted to an application of the two-port synthesis procedure. As this application, a resistively terminated two-port network with the transfer and driving-point admittances specified independently will be considered.

Factorization of N x N Matrices

If an $N \times N$ matrix $[r_{ij}]$ having polynomial elements can be successfully factored, the synthesis procedure developed in Chapter III can be applied to the N-port network with N controlled sources embedded in it. In some cases, det r_{ij} will have a sufficient number of distinct real zeros to ensure that the desired factorization can be accomplished by the technique of Appendix II. Even if det r_{ij} does not have a sufficient number of zeros to ensure success but has some real zeros, it may still be possible to accomplish the factorization, either partially or to the desired degree, by that technique. As in the case of 2×2 matrices, that technique should be used until the desired factorization is accomplished or until the technique is no longer applicable. If the technique of Appendix II fails before the desired number of matrices of linear elements has been factored out of $[r_{ij}]$, then a technique involving the complex zeros of det r_{ij} must be employed. It is the purpose of this section to consider the factorization of an $N \times N$ matrix by making use of the complex zeros of its determinant.

A straight-forward extension of the matrix factorization technique developed in Chapter III allows a pair of conjugate complex zeros of det r_{ij} to be used to reduce by one the degree of the elements of two columns of $[r_{ij}]$. The derivation of this extension will be given here, although, some of the details will be omitted since the reasoning is similar to that for the 2×2 matrix factorization technique of Chapter III.

For the derivation suppose that

1. $$[r_{ij}] = \begin{bmatrix} r_{11}(x) & r_{12}(x) & \cdots & r_{1N}(x) \\ r_{21}(x) & r_{22}(x) & \cdots & r_{2N}(x) \\ \cdot & \cdot & \cdot & \cdot \\ r_{N1}(x) & r_{N2}(x) & \cdots & r_{NN}(x) \end{bmatrix}$$

is the matrix to be factored.

2. the degree of r_{ij} is L.
3. $|r_{ij}| \not\equiv 0$.
4. x_1 and x_1^* are conjugate complex zeros of $|r_{ij}|$ and $\operatorname{Im} x_1 \neq 0$.
5. $\Delta_{11}(x_1) \neq 0$.
6. $\operatorname{Im} \dfrac{\Delta_{12}(x_1)}{\Delta_{11}(x_1)} \neq 0$.

In this derivation, $\Delta_{ij}(x_1)$ means the cofactor of the element $r_{ij}(x_1)$ of the matrix $[r_{ij}(x_1)]$. The corresponding minor will be designated $\Gamma_{ij}(x_1)$. Throughout the derivation $i,j = 1,2,\ldots,N$ unless otherwise stated.

Let

$$[H_{ij}] = \begin{bmatrix} a_{11}x - b_{11} & 0 & 0 & 0 & \ldots & 0 \\ a_{21}x - b_{21} & 1 & 0 & 0 & \ldots & 0 \\ a_{31}x - b_{31} & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{N1}x - b_{N1} & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}$$

where a_{i1} and b_{i1} are real constants. The first column of $[r_{ij}][H_{ij}]$ is the column matrix

$$[r_{ij}]\ H_{j1}] = [r_{ij}]\ a_{j1}]\ x - [r_{ij}]\ b_{j1}] \tag{89}$$

and any other column of $[r_{ij}][H_{ij}]$ is the same as the corresponding column of $[r_{ij}]$. Requiring each element of the first column of $[r_{ij}][H_{ij}]$ to be zero at $x = x_1$, Equation (89) becomes

$$0] = [r_{ij}(x_1)]\ a_{j1}]\ x_1 - [r_{ij}(x_1)]\ b_{j1}] \tag{90}$$

Equation (90) represents a system of N homogeneous equations with $2N$ unknowns a_{j1} and b_{j1}. The rank of the coefficient matrix is less than N. Hence, a nontrivial solution exists. The last $N - 1$ equations of that system of equations can be written as

$$0] = [r_{ij}(x_1)]\ a_{j1}]x_1 - [r_{ij}(x_1)]\ b_{j1}] \quad \text{for} \quad i \neq 1$$

$$= [r_{ij}(x_1)]a_{j1}]x_1 - [r_{ij}(x_1)]b_{j1}] + r_{i1}(x_1)](a_{11}x_1 - b_{11}) \text{ for } i,j \neq 1$$

Solving this equation for $b_{j1}]$ gives

$$b_{j1}] = [r_{ij}(x_1)]^{-1}\ r_{i1}(x_1)](a_{11}x_1 - b_{11}) + a_{j1}]x_1 \quad \text{for} \quad i,j \neq 1$$

$$= -\frac{\Delta_{1j}(x_1)}{\Delta_{11}(x_1)}(a_{11}x_1 - b_{11}) + a_{j1}]\ x_1 \quad \text{for} \quad j \neq 1 \tag{91}$$

For $N = 2$ Equation (91) reduces to Equation (48) in the 2 x 2 factorization discussion.

Choose a_{11} to be unity and b_{11} to be any real number other than a zero of det r_{ij}. Since Im $x_1 \neq 0$, each a_{j1} can be chosen such that b_{j1} is real for $j = 2,3,\ldots,N$. Then Equation (91) gives b_{j1}. Thus, a matrix $[H_{ij}]$ can be determined such that every element of the first column of the product $[r_{ij}][H_{ij}]$ contains the factors $(x - x_1)$ and $(x - x_1^*)$. That product can then be written as

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & r_{12} & r_{13} \cdots r_{1N} \\ g_{21} & r_{22} & r_{23} \cdots r_{2N} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g_{N1} & r_{N2} & r_{N3} \cdots r_{NN} \end{bmatrix} \begin{bmatrix} G & 0 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 & 0 \cdots 0 \\ 0 & 0 & 1 & 0 \cdots 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \cdots 1 \end{bmatrix} \tag{92}$$

where

$$g_{i1} = \frac{1}{G}\sum_{j=1}^{N} r_{ij}\ H_{j1}$$

and

$$G = (x - x_1)(x - x_1^*)$$

$$= x^2 - 2\text{Re } x_1 x + (\text{Re } x_1)^2 + (\text{Im } x_1)^2$$

The first matrix of the right-hand member of Equation (92) will now be considered. The determinant of this matrix has a zero at $x = b_{11}$. To remove the factor $(x - b_{11})$ from the second column of that matrix by means of the procedure of Appendix II requires that

$$\left[\begin{bmatrix} g_{11}(b_{11}) & r_{12}(b_{11}) & r_{13}(b_{11}) & \ldots & r_{1N}(b_{11}) \\ g_{21}(b_{11}) & r_{22}(b_{11}) & r_{23}(b_{11}) & \ldots & r_{2N}(b_{11}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{N1}(b_{11}) & r_{N2}(b_{11}) & r_{N3}(b_{11}) & \ldots & r_{NN}(b_{11}) \end{bmatrix} \begin{matrix} c_{12} \\ c_{22} \\ \\ c_{N2} \end{matrix}\right] = 0] \quad (93)$$

If one of the minors of the second column of the coefficient matrix of Equation (93) is nonzero, c_{22} can be chosen to be unity. Using the definition of g_{j1} and the fact that $H_{11}(b_{11}) = 0$, the minor of $r_{i2}(b_{11})$ becomes $H_{21}(b_{11})\,\Gamma_{i1}(b_{11})/G(b_{11})$. At least one of these minors, say $\Gamma_{11}(b_{11})$, is nonzero; otherwise, b_{11} would be a zero of det r_{ij}. It must now be shown that $H_{21}(b_{11}) \neq 0$. Using b_{21} as found from Equation (91), the defining equation for H_{21} gives

$$H_{21}(b_{11}) = a_{21}b_{11} - b_{21}$$

$$= a_{21}b_{11} + \frac{\Delta_{12}(x_1)}{\Delta_{11}(x_1)}(a_{11}x_1 - b_{11}) - a_{21}x_1$$

$$= \left(\frac{\Delta_{12}(x_1)}{\Delta_{11}(x_1)} - a_{21}\right)(x_1 - b_{11}) \quad (94)$$

Since $\mathrm{Im}\,[\Delta_{12}(x_1)/\Delta_{11}(x_1)] \neq 0$ and $\mathrm{Im}\, x_1 \neq 0$ and both a_{21} and b_{11} are real, Equation (94) shows that $H_{21}(b_{11}) \neq 0$. Hence, c_{22} of Equation (93) can be chosen to be unity. Making that choice and solving for the other constants of Equation (93) give

$$c_{12} = - \frac{\Delta_{11}(b_{11})G(b_{11})}{H_{21}(b_{11})\Delta_{11}(b_{11})} = - \frac{G(b_{11})}{H_{21}(b_{11})}$$

$$c_{i2} = \frac{H_{i1}(b_{11})\Delta_{11}(b_{11})}{H_{21}(b_{11})\Delta_{11}(b_{11})} = \frac{H_{i1}(b_{11})}{H_{21}(b_{11})} \tag{95}$$

for $i = 3,4,\ldots,N$. Thus, the nonsingular matrix

$$[c_{ij}] = \begin{bmatrix} 1 & c_{12} & 0 & 0 & 0 \ldots 0 \\ 0 & 1 & 0 & 0 & 0 \ldots 0 \\ 0 & c_{32} & 1 & 0 & 0 \ldots 0 \\ 0 & c_{42} & 0 & 1 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ 0 & c_{N2} & 0 & 0 & 0 \ldots 1 \end{bmatrix}$$

can be determined such that when the first matrix of the right-hand member of Equation (92) is postmultiplied by $[c_{ij}]$, every element of the second column of that product will contain the factor $(x - b_{11})$. Hence, Equation (92) can be written as

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & g_{12} & r_{13} \ldots r_{1N} \\ g_{21} & g_{22} & r_{23} \ldots r_{2N} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ g_{N1} & g_{N2} & r_{N3} \ldots r_{NN} \end{bmatrix} \begin{bmatrix} G & -c_{12} & 0 & 0 \ldots 0 \\ 0 & H_{11} & 0 & 0 \ldots 0 \\ 0 & -c_{32} & 1 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ 0 & -c_{N2} & 0 & 0 \ldots 1 \end{bmatrix} \tag{96}$$

where

$$g_{i2} = \frac{c_{12}g_{i1} + \sum_{j=2}^{N} c_{j2}r_{ij}}{H_{11}}$$

is a first degree polynomial.

Next, the last matrix of the right-hand member of Equation (96) will be considered. The determinant of this matrix has a zero at $x = b_{11}$. Using that zero and the procedure of Appendix II to reduce the degree of the element of the first column of the matrix under consideration requires that

$$\begin{bmatrix} \begin{bmatrix} G(b_{11}) & -c_{12} & 0 & 0 \ldots 0 \\ 0 & H_{11}(b_{11}) & 0 & 0 \ldots 0 \\ 0 & -c_{32} & 1 & 0 \ldots 0 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ 0 & -c_{N2} & 0 & 0 \ldots 1 \end{bmatrix} & \begin{matrix} d_{11} \\ d_{21} \\ d_{31} \\ \vdots \\ d_{N1} \end{matrix} \end{bmatrix} = 0 \qquad (97)$$

Since $G(b_{11}) \neq 0$, Equations (95) show that $c_{12} \neq 0$. With $c_{12} \neq 0$, it can be seen that the minor of the $2,1$ element of the coefficient matrix of Equation (97) is nonzero. This allows d_{11} to be chosen nonzero. Making the choice $d_{11} = 1$, solving for the other unknowns of Equation (97), and using Equations (95) give

$$\begin{aligned} d_{21} &= \frac{G(b_{11})}{c_{12}} \\ &= - H_{21}(b_{11}) \end{aligned} \qquad (98)$$

and

$$d_{i1} = \frac{G(b_{11})c_{i2}}{c_{12}}$$

$$= \frac{G(b_{11})H_{21}(b_{11})H_{i1}(b_{11})}{-G(b_{11})H_{21}(b_{11})}$$

$$= - H_{i1}(b_{11}) \tag{99}$$

for $i = 2,3,\ldots,N$.

Equation (96) can now be written as

$$[r_{ij}][H_{ij}] = \begin{bmatrix} g_{11} & g_{12} & r_{13} & \cdots & r_{1N} \\ g_{21} & g_{22} & r_{23} & \cdots & r_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ g_{N1} & g_{N2} & r_{N3} & \cdots & r_{NN} \end{bmatrix}$$

$$\begin{bmatrix} k_{11} & -c_{12} & 0 & 0 & \cdots & 0 \\ d_{21} & H_{11} & 0 & 0 & \cdots & 0 \\ 0 & -c_{32} & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -c_{N2} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} H_{11} & 0 & 0 & 0 & \cdots & 0 \\ -d_{21} & 1 & 0 & 0 & \cdots & 0 \\ -d_{31} & 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -d_{N1} & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{100}$$

where

$$k_{11} = \frac{G - c_{12}d_{21}}{H_{11}}$$

is a first degree polynomial.

As in the case of 2×2 matrices, the right-hand member of Equation (100) can be modified so that the last matrix of that member is $[H_{ij}]$. To accomplish this, the last matrix is premultiplied by $[a_{ij}]$ and the second matrix is postmultiplied by $[a_{ij}]^{-1}$ where

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \ldots 0 \\ a_{21} & 1 & 0 & 0 \ldots 0 \\ a_{31} & 0 & 1 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_{N1} & 0 & 0 & 0 \ldots 1 \end{bmatrix}$$

and the a_{i1} have been determined previously for $i = 2,3,\ldots,N$. Premultiplying the last matrix of the right-hand member of Equation (100) by $[a_{ij}]$ gives a matrix whose N diagonal elements are $H_{11},1,1,\ldots,1$. Writing the $i,1$ element of this product for $i \neq 1$ and using Equations (98) and (99) along with the definition of H_{i1} give

$$\begin{aligned} a_{i1}H_{11} - d_{i1} &= a_{i1}(x - b_{11}) - H_{i1}(b_{11}) \\ &= a_{i1}x - b_{i1} = H_{i1} \end{aligned}$$

The nondiagonal elements of all columns except the first are zero. Thus, the product in question is $[H_{ij}]$.

After inserting $[a_{ij}]$ and $[a_{ij}]^{-1}$ in Equation (100) as explained in the above paragraph, that equation can be postmultiplied by $[H_{ij}]^{-1}$. This gives

$$[r_{ij}] = \begin{bmatrix} g_{11} & g_{12} & r_{13} \cdots r_{1N} \\ g_{21} & g_{22} & r_{23} \cdots r_{2N} \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ g_{N1} & g_{N2} & r_{N3} \cdots r_{NN} \end{bmatrix} \begin{bmatrix} a_{21}c_{12} + h_{11} & -c_{12} & 0 & 0 \ldots 0 \\ -H_{21} & H_{11} & 0 & 0 \ldots 0 \\ a_{21}c_{32} - a_{31} & -c_{32} & 1 & 0 \ldots 0 \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ a_{21}c_{N2} - a_{N1} & -c_{N2} & 0 & 0 \ldots 1 \end{bmatrix} \quad (101)$$

where $d_{21} - a_{21}H_{11}$ has been replaced by its equivalent $-H_{21}$ as the 2,1 element of the last factor. All of the symbols of Equation (101) were defined earlier in the derivation. Reference to these definitions show that g_{i1} and g_{i2} have degree $L - 1$. In the second factor of Equation (101), the elements are either first degree polynomials or constants. Furthermore, the first degree elements are confined to the 2×2 submatrix of the upper left-hand corner.

In the preceding derivation, columns 1 and 2 specifically were chosen to be the columns whose elements were to be reduced in degree. There is no loss of generality in making this choice because a given matrix can be postmultiplied by a nonsingular matrix of constants to bring any desired columns into the first and second positions. After reducing the degree of the elements of columns 1 and 2 of the modified matrix, the inverse operation is performed on the columns and the rows of the second factor and on the columns of the first factor.

As in the case of 2×2 matrices, there is no assurance that the procedure developed in this section can be repeated the desired number of times. However, the sufficient conditions for carrying out the procedure are so lenient that this procedure is certainly a useful tool for the problem of factoring $N \times N$ matrices.

The Resistively Terminated Two-port

Let a lossless two-port network with two controlled sources embedded in it be terminated in a one ohm resistor as shown in Figure 7.

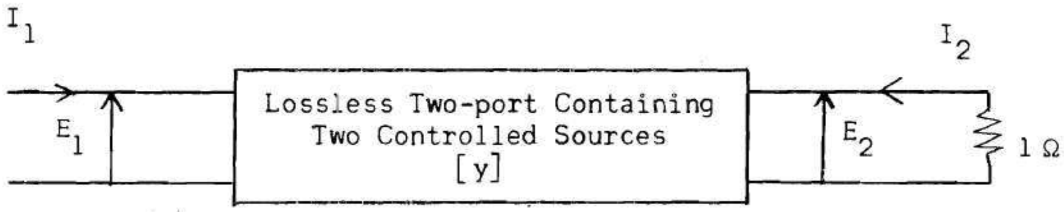


Figure 7. A Resistively Terminated Two-port Network - An Application of the Two-port Synthesis Procedure.

For the network of Figure 7, the driving-point admittance is

$$Y_{11} = \frac{I_1}{E_1} = y_{11} - \frac{y_{12}y_{21}}{y_{22}+1} \tag{102}$$

and the transfer admittance is

$$Y_{21} = \frac{I_2}{E_1} = \frac{y_{21}}{y_{22}+1} \tag{103}$$

Let the prescribed driving-point admittance be

$$\tilde{Y}_{11} = \frac{M_{11} + N_{11}}{M + N} \tag{104}$$

and the prescribed transfer admittance be

$$\tilde{Y}_{21} = \frac{N_{21}}{M_2 + N_2} \tag{105}$$

where M, M_{11} and M_2 are even polynomials and N, N_{11}, N_2 and N_{21} are odd polynomials in s. The problem is to find an admittance matrix $[y]$ such that the driving-point and transfer admittances given by Equations (102) and (103) are equal to those prescribed by Equations (104) and (105) respectively and such that $[y]$ is realizable by the two-port synthesis procedure of Chapter III.

One possible identification of the admittance parameters can be obtained by rearranging Equation (105) as

$$\tilde{Y}_{21} = \frac{N_{21}}{M_2 + N_2} = \frac{\dfrac{\rho N_{21}}{M + N}}{\dfrac{\rho(M_2 + N_2)}{M + N}}$$

$$= \frac{\dfrac{\rho\, N_{21}}{M + N}}{\dfrac{\rho(M_2 + N_2) - (M + N)}{M + N} + 1} \tag{106}$$

where ρ is an arbitrary constant. From the right-hand members of Equations (103) and (106), the following identifications can be made:

$$y_{21} = \frac{\rho\, N_{21}}{M + N} \tag{107}$$

$$y_{22} = \frac{\rho(M_2 + N_2) - (M + N)}{M + N} \tag{108}$$

Let

$$y_{12} = 0$$

Then from Equations (102) and (105)

$$y_{11} = Y_{11} = \frac{M_{11} + N_{11}}{M + N} \tag{110}$$

The synthesis procedure of Chapter III requires that $[\overline{y}]_{11}$ be realizable as a lossless network. Using the parameters defined by Equations (107), (108), (109) and (110), the matrix $[\overline{y}]_{11}$ for Case B is nonsymmetric, and thus not realizable as a lossless network. However, for Case A that matrix becomes

$$[\overline{y}]_{11} = \frac{[m_{ij}]}{n} = \frac{1}{N}\begin{bmatrix} M_{11} & 0 \\ 0 & \rho M_2 - M \end{bmatrix}$$

If M_2 has only simple j-axis zeros and deg $M_2 \geq$ deg M, ρ can be chosen large enough so that every zero of $\rho M_2 - M$ lies on the j-axis and one of those zeros is arbitrarily close to each zero of M_2. Thus, if M_2/N is a reactance function with M_2 and N having no common factors, ρ can be chosen large enough so that $(\rho M_2 - M)/N$ will be a reactance function. If, in addition, M_{11}/N is a reactance function with M_{11} and N having no common factors, the matrix $[\overline{y}]_{11}$ will be realizable as a lossless network and further, the residue matrix at every finite pole will be positive definite. Now, if deg M_{11} = deg M_2, the residue matrix of the possible pole at infinity will be positive definite also. Hence, Conditions 1 under Case A of Theorem 3 will be satisfied.

The function M_{11}/N being a reactance function with M_{11} and N having no common factors ensures that Condition 2 of Theorem 3 is satisfied.

The matrix from which $[\overline{y}]_{13}$ and $[\overline{y}]_{21}$ are found is

$$[nn_{ij} - mm_{ij}] = \begin{bmatrix} NN_{11} - MM_{11} & 0 \\ NN_{21} & N(\rho N_2 - N) - M(\rho M_2 - M) \end{bmatrix}$$

If this matrix can be factored so as to give satisfactory submatrices $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$, then Condition 3 of Theorem 3 will be satisfied.

The Case B synthesis procedure is suggested if the transfer admittance is given by

$$\tilde{Y}_{21} = \frac{M_{21}}{M_2 + N_2}$$

where M_{21} is an even polynomial. The identification scheme follows the same pattern as for Case A.

From the foregoing discussion, the following theorem can be stated:

Theorem 4

Let the prescribed driving-point admittance of the resistively terminated network of Figure 7 be

$$\tilde{Y}_{11} = \frac{M_{11} + N_{11}}{M + N}$$

and let the prescribed transfer admittance be

$$\tilde{Y}_{21} = \frac{N_{21}}{M_2 + N_2} \quad \text{for Case A}$$

$$\tilde{Y}_{21} = \frac{M_{21}}{M_2 + N_2} \quad \text{for Case B}$$

where M, M_{11}, M_2 and M_{21} are even polynomials and N, N_{11}, N_2 and N_{21} are odd polynomials in s. The two-port network of that figure is realizable as a lossless network with two current-controlled voltage sources embedded in it if the conditions under Case A or Case B below are satisfied:

<u>Case A</u>

1. M_{11}/N is a reactance function with M_{11} and N having no common factors.

2. M_2/N is a reactance function with M_2 and N having no common factors.

3. Deg M_{11} = deg $M_2 \geq$ deg M.

4. After choosing ρ large enough so that $(\rho M - M)/N$ is a reactance function, it is possible to factor the matrix

$$\begin{bmatrix} NN_{11} - MM_{11} & 0 \\ NN_{21} & N(\rho N_2 - N) - M(\rho M_2 - M) \end{bmatrix}$$

so that $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ are satisfactory submatrices.

<u>Case B</u>

1. N_{11}/M is a reactance function with N_{11} and M having no common factors.

2. N_2/M is a reactance function with N_2 and M having no common factors.

3. Deg N_{11} = deg $N_2 \geq$ deg N.

4. After choosing ρ large enough so that $(\rho N_2 - N)/M$ is a reactance function, it is possible to factor the matrix

$$\begin{bmatrix} MM_{11} - NN_{11} & 0 \\ MM_{21} & M(\rho M_2 - M) - N(\rho N_2 - N) \end{bmatrix}$$

so that $[\bar{y}]_{13}$ and $[\bar{y}]_{21}$ are satisfactory submatrices.

To illustrate the procedure presented in this section, suppose that the prescribed driving-point admittance for the network of Figure 7 is

$$\tilde{Y}_{11} = \frac{M_{11} + N_{11}}{M + N} = \frac{s^2 + 5s + 1}{s^2 + s + 2} \tag{111}$$

and that the prescribed transfer admittance is

$$\tilde{Y}_{21} = \frac{N_{21}}{M + N} = \frac{10s}{s^2 + 2s + 3} \tag{112}$$

Case A is indicated since $\tilde{Y}_{21}$ has the proper form for that case and also Conditions 1, 2 and 3 under Case A of Theorem 4 are satisfied. For this case, ρ must be chosen such that

$$\frac{\rho M_2 - M}{N} = \frac{\rho(s^2 + 3) - (s^2 + 2)}{s}$$

is a reactance function. The choice $\rho = 2$ gives

$$\frac{\rho M_2 - M}{N} = \frac{s^2 + 4}{s}$$

which is reactance as desired. With this choice of ρ, Equations (107), (108), (109) and (110) give the admittance parameters for the two-port network of Figure 7. The admittance matrix is

$$[y] = \frac{1}{s^2 + s + 2}\begin{bmatrix} s^2 + 5s + 1 & 0 \\ 20s & s^2 + 3s + 4 \end{bmatrix} \tag{113}$$

The synthesis procedure of Chapter III is now used to realize Equation (113) as a lossless two-port network with two current-controlled voltage sources embedded in it. Equations (33), (40) and (38) give

$$[\bar{y}]_{11} = \frac{[m_{ij}]}{n} = \frac{1}{s}\begin{bmatrix} s^2 + 1 & 0 \\ 0 & s^2 + 4 \end{bmatrix} \tag{114}$$

$$[R][\bar{y}]_{23} = -\frac{m}{n}[U] = \frac{1}{s}\begin{bmatrix} -s^2 - 2 & 0 \\ 0 & -s^2 - 2 \end{bmatrix} \tag{115}$$

$$[\bar{y}]_{13}[R]_1[R]_2[\bar{y}]_{21} = \frac{[nn_{ij} - mm_{ij}]}{n^2}$$

$$= \frac{1}{s}\begin{bmatrix} -s^4 + 2s - 2 & 0 \\ 20s^2 & -s^4 - 3s^2 - 8 \end{bmatrix} \tag{116}$$

The matrix of the right-hand member of Equation (116) must be factored. Replacing s^2 by x and designating the resulting matrix by $[r_{ij}]$ gives

$$[r_{ij}] = \begin{bmatrix} -s^2 + 2x - 2 & 0 \\ 20x & -x^2 - 3x - 8 \end{bmatrix}$$

from which

$$|r_{ij}| = (-x^2 + 2x - 2)(-x^2 - 3x - 8)$$

$$= (x - 1 - j1)(x - 1 + j1)(x^2 + 3x + 8)$$

The matrix factorization technique of Chapter III can be used to remove a matrix of linear elements from $[r_{ij}]$. Using the pair of conjugate complex zeros x_1, $x_1^* = 1 \pm j1$ and choosing $b_{11} = 1$, that factorization procedure gives

$$[r_{ij}] = \begin{bmatrix} -x + 1 & -\frac{73}{60} \\ -\frac{160}{73}x + \frac{880}{73} & -x - \frac{20}{3} \end{bmatrix} \begin{bmatrix} x + \frac{5}{3} & -\frac{73}{60} \\ -\frac{160}{73}x + \frac{220}{73} & x - 1 \end{bmatrix}$$

Having the factors of $[r_{ij}]$, it can be seen that Equation (116) is satisfied when

$$[\bar{y}]_{13} = \frac{1}{s} \begin{bmatrix} -s^2 + 1 & -\frac{73}{60} \\ -\frac{160}{73}s^2 + \frac{880}{73} & -s^2 - \frac{20}{3} \end{bmatrix} [R]_1^{-1} \tag{117}$$

and

$$[y]_{21} = \frac{1}{s}[R]_2^{-1} \begin{bmatrix} s^2 + \frac{5}{3} & -\frac{73}{60} \\ -\frac{160}{73}s^2 + \frac{220}{73} & s^2 - 1 \end{bmatrix} \tag{118}$$

Choosing

$$[R]_1 = \begin{bmatrix} \frac{400}{73} & 0 \\ 0 & \frac{400}{60} \end{bmatrix} \quad [R]_2 = \begin{bmatrix} \frac{400}{60} & 0 \\ 0 & \frac{400}{73} \end{bmatrix} \tag{119}$$

Equations (117) and (118) become

$$[\bar{y}]_{13} = \frac{1}{400s} \begin{bmatrix} -73s^2 + 73 & -73 \\ -160s^2 + 880 & -60s^2 - 400 \end{bmatrix} \tag{120}$$

and

$$[\bar{y}]_{21} = \frac{1}{400s} \begin{bmatrix} 60s^2 + 100 & -73 \\ -160s^2 + 220 & 73s^2 - 73 \end{bmatrix} \tag{121}$$

With the choices indicated in Equations (119), the coefficient matrix is

$$[R] = [R]_1[R]_2 = \begin{bmatrix} \frac{8000}{219} & 0 \\ 0 & \frac{8000}{219} \end{bmatrix}$$

From this matrix and Equation (115)

$$[\bar{y}]_{23} = \frac{1}{400s} \begin{bmatrix} -\frac{219}{20}(s^2 + 2) & 0 \\ 0 & -\frac{219}{20}(s^2 + 2) \end{bmatrix} \tag{122}$$

Using Equations (114), (120), (121) and (122), the admittance matrix for the lossless six-port network becomes

$$[\bar{y}] = \frac{s}{400}\begin{bmatrix} 400 & 0 & 60 & -160 & -73 & 0 \\ 0 & 400 & 0 & 73 & -160 & -60 \\ 60 & 0 & k_{33}^{(\infty)} & 0 & -10.85 & 0 \\ -160 & 73 & 0 & k_{44}^{(\infty)} & 0 & -10.85 \\ -73 & -160 & -10.85 & 0 & k_{55}^{(\infty)} & 0 \\ 0 & -60 & 0 & -10.85 & 0 & k_{66}^{(\infty)} \end{bmatrix}$$

$$+ \frac{1}{400s}\begin{bmatrix} 400 & 0 & 100 & 220 & 73 & -73 \\ 0 & 1600 & -73 & -73 & 880 & -400 \\ 100 & -73 & k_{33}^{(0)} & 0 & -21.7 & 0 \\ 220 & -73 & 0 & k_{44}^{(0)} & 0 & -21.7 \\ 73 & 880 & -21.7 & 0 & k_{55}^{(0)} & 0 \\ -73 & -400 & 0 & -21.7 & 0 & k_{66}^{(0)} \end{bmatrix} \qquad (123)$$

The first term of the right-hand member of this equation is realizable as a network of capacitors and the second term as a network of inductors. Connecting these in parallel gives the lossless six-port network.

The coefficients of the controlled sources are the elements of $[R]$. Connecting these sources to the appropriate ports of the lossless six-port network gives a two-port realization of Equation (113). Terminating this two-port network in a one-ohm resistor as indicated by Figure 8 gives a realization of the driving-point and transfer admittance of Equations (111) and (112) respectively.

Summary

In the first section of this chapter, a technique useful in the factorization of an N x N matrix $[r_{ij}]$ having polynomial elements was

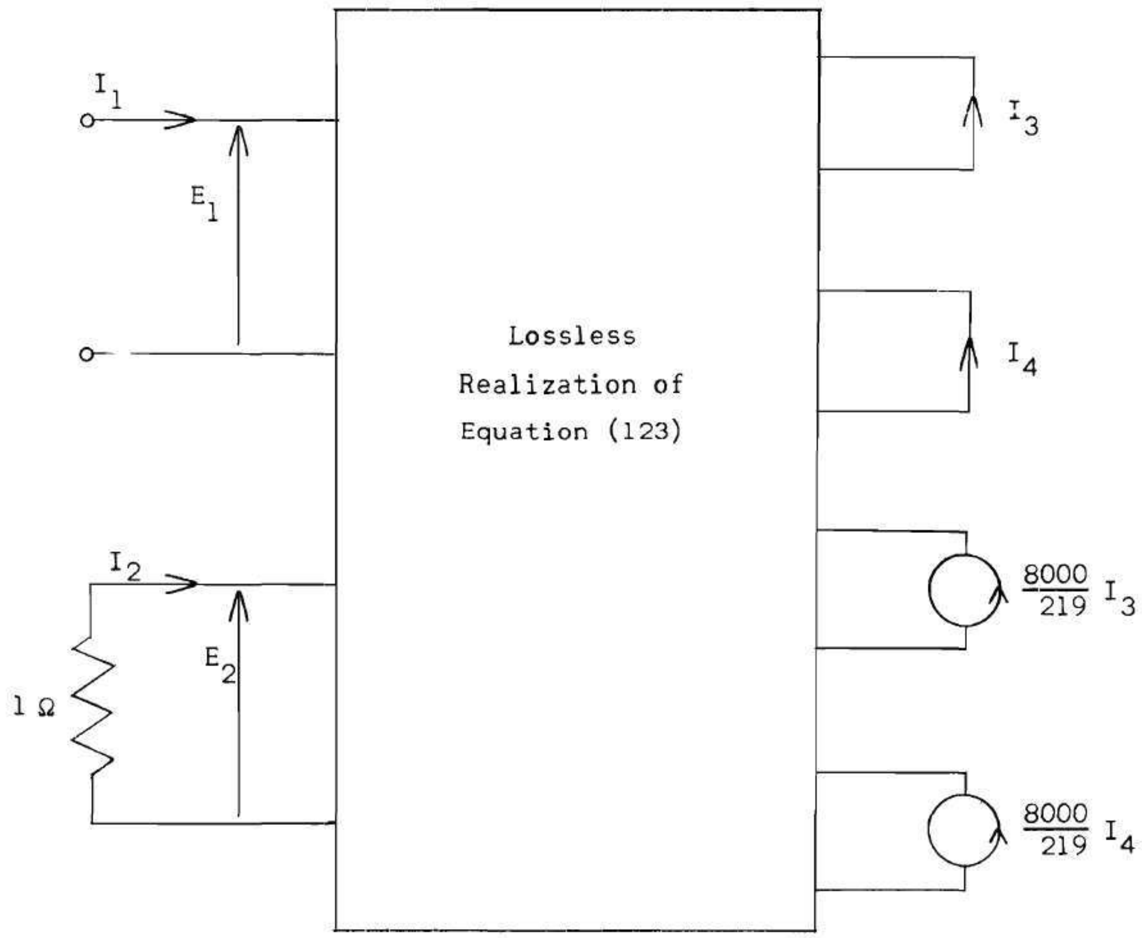


Figure 8. Resistively Terminated Two-port Network Realizing the Driving-point and Transfer Admittances Given by Equations (111) and (112) respectively.

developed. This procedure makes use of a pair of conjugate complex zeros of det r_{ij} to reduce the degree of the elements of two columns of $[r_{ij}]$. The technique is an extension of that developed for 2×2 matrices in Chapter III.

The two-port network terminated in a resistor was considered in the second section. A procedure and some sufficient conditions were given for using the synthesis procedure of Chapter III for finding a two-port network with two current-controlled voltage sources embedded in it when the transfer and driving-point admittances for the terminated network were prescribed. An example using this procedure was given.

CHAPTER V

CONCLUSIONS

This investigation has resulted in some useful synthesis procedures for lossless networks with controlled sources embedded in them. In this chapter, some remarks will be made regarding the solutions offered in this thesis.

For the lossless one-port network with one current-controlled voltage source embedded in it, the sufficient conditions which are given in Theorem 2 ensure that the one-port synthesis procedure will be successful. It should be noted that these conditions are not necessary. For example, the function

$$\tilde{y} = \frac{0.1s^3 - 8s^2 + 1.5s + 5}{s^3 + 14s - 1}$$

does not satisfy the conditions under either Case A or Case B of Theorem 2. However, multiplication of both numerator and denominator by the augmenting function

$$m_o + n_o = 10s + 1$$

gives

$$\tilde{y} = \frac{0.1s^3 - 8s^2 + 1.5s + 5}{s^3 + 14s - 1} \quad \frac{10s + 1}{10s + 1}$$

$$= \frac{s^4 - 79.9s^3 + 7s^2 + 51.5s + 5}{10s^4 + s^3 + 140s^2 + 4s - 1}$$

which does satisfy the conditions under Case A of Theorem 2. Thus, $\tilde{y}$ is realizable as a driving-point admittance of a lossless network with one current-controlled voltage source embedded in it. The problem of finding the augmenting function $m_o + n_o$ to make the prescribed function realizable by a network of this class has not been solved.

A synthesis procedure was developed for the lossless two-port network with two current-controlled voltage sources embedded in it. For use with this procedure a technique was developed for factoring a 2×2 matrix of polynomial elements when the determinant of that matrix has complex zeros. In the development of this technique, sufficient conditions were given (in the hypothesis) for the removal of a matrix of linear elements. This removal leaves a matrix having elements one degree lower than those of the original matrix. One shortcoming of the technique is that there is no assurance that the reduced matrix will satisfy the conditions ensuring that a second matrix of linear elements can be removed. In other words, no conditions on the given matrix have been found to ensure that the factorization technique can be repeated the desired number of times. This does not appear to be a serious shortcoming because the sufficient conditions for the removal of a matrix of linear elements are very lenient.

The application of the synthesis procedure of Chapter III to the N-port case requires the factorization of an $N \times N$ matrix $[r_{ij}]$ having polynomial elements. It was shown in Chapter IV that if $[r_{ij}]$ satisfied certain conditions, then a pair of conjugate complex zeros of det r_{ij} could be used to reduce by one the degree of the elements of two columns of $[r_{ij}]$. Attempts to determine the exact conditions on

$[r_{ij}]$ under which the technique could be applied to all columns were not successful. Again, however, the sufficient conditions for the reduction of the degree of the elements of two columns are very lenient. Hence, it appears that most matrices will allow the technique to be repeated so long as the determinant has a pair of conjugate complex zeros. Additional work on the problem of factoring $N \times N$ matrices whose determinants have complex zeros seems desirable.

The application of the N-port synthesis procedure for $N = 1$ gives the same identification for the lossless network parameters as the procedure of Chapter II. In this case, of course, the factorization of the $N \times N$ matrix becomes simply the factorization of a polynomial.

The two-port synthesis procedure can be used in the problem of finding a two-port network such that when that network is terminated in a resistor, it will have a prescribed transfer function and a prescribed driving-point admittance. To ensure success, the prescribed transfer and driving-point admittances must satisfy the conditions of Theorem 4 but can be otherwise arbitrary. This allows the transfer admittance to be prescribed with a large multiplicative constant and thus allows a power gain from port 1 to the terminating resistor. Also, it allows the driving-point admittance to be prescribed independently, to some extent, of the transfer admittance.

APPENDIX I

NECESSARY AND SUFFICIENT CONDITIONS FOR A LOSSLESS NETWORK

The synthesis procedures developed in this thesis and the proofs of the theorems depend on the necessary and sufficient conditions for lossless networks. For the convenience of the reader, these conditions are given in this appendix. The theorem which is stated below is Theorem 7-12 in a book by Weinberg (8).

Theorem

A symmetric N^{th}-order matrix of real rational functions is realizable as the $[\overline{y}]$ of an LC network (containing ideal transformers) if and only if it satisfies the three conditions:

1. For $i = 1,2,\ldots,N$, $\overline{y}_{ii}$ is an F_{LC} function.

2. Each of the matrix elements $\overline{y}_{ij}$ for $i \neq j$ is a nonconstant function that has only simple poles on the j axis with real residues.

3. The matrix of residues at each pole s_v (including a possible pole at infinity) is positive semidefinite.

In the above theorem an F_{LC} function means a reactance function; i.e., one that is realizable as the driving-point admittance of an LC network.

In regard to Condition 2, it should be noted that for $\overline{y}_{ij}$ the real residues may have zero value at every pole. Thus, $\overline{y}_{ij} \equiv 0$ for $i \neq j$ does not render the $[\overline{y}]$ unrealizable. Condition 2 may be replaced by Condition 2a, which may be more convenient for some uses.

2a. Each of the matrix elements $\overline{y}_{ij}$ for $i \neq j$ is an odd function that has only j axis poles and those are simple.

APPENDIX II

A MATRIX FACTORIZATION TECHNIQUE FOR USE WHEN THE DETERMINANT HAS REAL ZEROS

In this appendix a technique will be given for factoring an $N \times N$ matrix of polynomial elements when the determinant of that matrix has a sufficient number of distinct real zeros. The development of this technique can be found in papers by Sandberg (6), (7) and in a book by Su (1). Throughout this appendix $i, j = 1,2,\ldots,N$ unless otherwise stated.

Suppose that

1. $[r_{ij}]$ is the matrix to be factored.
2. r_{ij} are polynomials in x with degree L.
3. $|r_{ij}| \not\equiv 0$.
4. $x_1, x_2,\ldots,x_k$ are k distinct real zeros of $|r_{ij}|$.

The first step in the procedure is to determine a nonsingular real matrix

$$[c_v] = \begin{bmatrix} 1 & & c_{1v} & & \\ & \ddots & \vdots & & \\ & & c_{vv} & & \\ & & \vdots & \ddots & \\ & & c_{Nv} & & 1 \end{bmatrix}$$

such that the product $[r_{ij}][c_v]$ has one of the linear factors of $|r_{ij}|$, say $(x - x_v)$, in every element of the v^{th} column. If that product is defined by

$$[r_{ij}][c_v] = [\hat{r}_{ij}] \ ,$$

then

$$\hat{r}_{ij} = r_{ij} \quad \text{for} \quad j \neq v$$

and

$$r_{iv} = r_{i1}c_{1v} + r_{i2}c_{2v} + \ldots + r_{iN}c_{Nv}$$

with r_{iv} containing the factor $(x - x_1)$. An important feature in this procedure is that only the v^{th} column is modified, leaving all other columns unchanged.

To determine the matrix $[c_v]$, the following system of homogeneous linear equations must be solved for the unknowns $c_{1v}, c_{2v}, \ldots, c_{Nv}$ and furthermore c_{vv} must be nonzero:

$$\begin{aligned} r_{11}(x_v)c_{1v} + r_{12}(x_v)c_{2v} + \ldots + r_{1N}(x_v)c_{Nv} &= 0 \\ r_{21}(x_v)c_{1v} + r_{22}(x_v)c_{2v} + \ldots + r_{2N}(x_v)c_{Nv} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ r_{N1}(x_v)c_{1v} + r_{N2}(x_v)c_{2v} + \ldots + r_{NN}(x_v)c_{Nv} &= 0 \end{aligned} \tag{A1}$$

It has been shown (1), (6), (7) that if the inequality

$$k > L(N - 1) \tag{A2}$$

is satisfied, then there exists a zero of $|r_{ij}|$, call it x_v, for which the system of Equations (A1) has a solution with $c_{vv} \neq 0$. Choosing $c_{vv} = 1$ and solving Equations (A1) gives the required nonsingular matrix $[c_v]$.

Since the zeros of $|[r_{ij}][c_v]|$ are identical to those of $|r_{ij}|$, the Inequality (A2) ensures that the above procedure can be applied to the product $[r_{ij}][c_v]$ for some new value of v, say u. Furthermore, applying that procedure to the matrix $[r_{ij}][c_v]$ will change only the u^{th} column and will not destroy the work of the previous step (or steps). Thus, if the above procedure is performed successively for $v = 1,2,\ldots,N$, every element of the v^{th} column of the product $[r_{ij}][c_1][c_2]\ldots[c_N]$ will contain the factor $(x - x_v)$ for $v = 1,2,\ldots,N$. Then, the product can be written as

$$[r_{ij}][c_1][c_2]\ldots[c_N] = [g_{ij}] \begin{bmatrix} (x - x_1) & & & \\ & (x - x_2) & & \\ & & \ddots & \\ & & & (x - x_N) \end{bmatrix} \quad \text{(A3)}$$

where the elements of $[g_{ij}]$ are the same as those of the product $[r_{ij}][c_1][c_2]\ldots[c_N]$ except for the factor $(x - x_v)$, which has been removed from the v^{th} column of that product. Clearly, the degree of g_{ij} is $L - 1$ or less. Defining $[h_{ij}]$ by

$$[h_{ij}] = \begin{bmatrix} (x - x_1) & & & \\ & (x - x_2) & & \\ & & \ddots & \\ & & & (x - x_N) \end{bmatrix} [c_N]\ldots[c_2]^{-1}[c_1]^{-1} \quad \text{(A4)}$$

Equation (A3) becomes

$$[r_{ij}] = [g_{ij}][h_{ij}]$$

From Equation (A4) it can be seen that the degree of h_{ij} is one or less.

If it is desired to remove k factors such as $[h_{ij}]$, then the Inequality (A2) must be satisfied (to ensure success) after the removal of the $(k - 1)^{th}$ factor. This requires that

$$k - N(k - 1) > (L - k + 1)(N - 1)$$

from which

$$k > L(N - 1) + k - 1 \qquad \text{(A5)}$$

It should be emphasized that Inequality (A5), as well as (A2), is a sufficient condition only; i.e., it ensures that k matrices having linear elements can be removed from $[r_{ij}]$ leaving a matrix having elements of degree $L - k$ or less. Usually the factorization procedure described in this appendix is applicable, even when the Inequality (A5) is not satisfied. In order to remove k factors such as $[h_{ij}]$ it is necessary that the number of real zeros of $|r_{ij}|$ be equal to or greater than kN.

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VITA

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